Funkcije - neprekidnost

March 13, 2022

Funkcije - neprekidnost

March 13, 2022

Funkcija $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$, je **neprekidna u tački** $x_0 \in X$ akko

$$(\forall \varepsilon > 0)(\exists \delta(\varepsilon) > 0)\big(\forall x \in X\big)\big(|x - x_0| < \delta(\varepsilon) \Rightarrow |f(x) - f(x_0)| < \varepsilon\big).$$

Funkcija $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$, je **neprekidna u tački $x_0 \in X$ sa desne (tj. leve)** strane ako je neprekidna za sve x iz skupa $X \cap (x_0, \infty)$ (tj. $X \cap (-\infty, x_0)$).

Razlike između definicija granične vrednosti i neprekidnosti u tački $x_0 \in X$ su:

- za graničnu vrednost u tački x_0 pretpostavka je da je x_0 tačka nagomilavanja za X, a kod neprekidnosti da je funkcija f definisana u tački x_0, tj. da $x_0 \in X$;
- kod neprekidnosti se zahteva da za svako $x \in X$ važi $|x - x_0| < \delta(\varepsilon) \Rightarrow |f(x) - f(x_0)| < \varepsilon$, a kod granične vrednosti da za svako $x \in X \setminus \{x_0\}$ važi $|x - x_0| < \delta(\varepsilon) \Rightarrow |f(x) - a| < \varepsilon$.

Odavde se može zaključiti sledeće:

- ako je f neprekidna funkcija u tački x_0 ne mora da postoji $\lim\limits_{x\longrightarrow x_0} f(x)$, jer ako je x_0 izolovana tačka za skup X, tada je f automatski neprekidna u tački x_0, dok u tom slučaju ne postoji granična vrednost $\lim\limits_{x\longrightarrow x_0} f(x)$;
- ako postoji $\lim\limits_{x\longrightarrow x_0} f(x)$ bez obzira da li je funkcija f definisana u tački x_0, funkcija ne mora da bude neprekidna u tački x_0 (na primer za $f(x) = \frac{\sin x}{x}$ je $\lim\limits_{x\longrightarrow 1} \frac{\sin x}{x} = 1$, a $f(x)$ nije neprekidna u 0 jer nije definisana u 0).

Dakle, da bi funkcija bila neprekidna u tački x_0 treba da važi:

- $x_0 \in X$, tj. funkcija je definisana u tački x_0;
- ako je x_0 tačka nagomilavanja za X, tada postoji $\lim\limits_{x \longrightarrow x_0} f(x)$ i važi jednakost $\lim\limits_{x \longrightarrow x_0} f(x) = f(x_0)$;
- ako je $x_0 \in X$ izolovana tačka, tada je f neprekidna u tački x_0 (ali nema graničnu vrednost u toj tački).

Ako je tačka x_0 tačka nagomilavanja skupa X i ako je funkcija f definisana u njoj, tada je f neprekidna u tački x_0 akko

$$\boxed{\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \boxed{\lim_{x \to x_0} f(x) = f(x_0).}}$$

Ako funkcija f nije neprekidna u tački x_0, kaže se da je funkcija f prekidna u tački x_0, odnosno da funkcija f ima prekid u tački x_0.

Funkcija je neprekidna nad skupom ako je neprekidna u svakoj tački tog skupa.

Elementarne funkcije su neprekidne na svojim domenima.

Ako su funkcije f i g neprekidne, tada su neprekidne i funkcije:

$$f \pm g,\ f \cdot g,\ \frac{f}{g},\ g \neq 0,\ \text{i } f \circ g$$

nad odgovarajućim domenima.

Neka su date funkcije $f : Y \to Z$ i $g : X \to Y$. Ako je $\lim\limits_{x \to x_0} g(x) = A$ i f neprekidna u tački A, tada je:

$$\lim_{x \to x_0} (f(g(x)) = f(\lim_{x \to x_0} g(x)) = f(A).$$

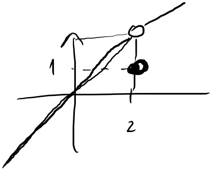
ZADACI

1. Ispitati neprekidnost sledećih funkcija:

$$1.1 \quad f(x) = \begin{cases} x, & x \neq 2 \\ 1, & x = 2 \end{cases}, \text{ u tački } x = 2;$$

$$\lim_{x \to 2} f(x) = \underbrace{f(2)}_{1}$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x = 2$$



$$\lim_{x \to 2} f(x) = f(2)$$

$$f(2) = 1 \qquad \neq$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x = 2$$

NIJE NEPREKIDNA

1.2 $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, tački $x = 0$;

$$\lim_{x\to 0} f(x) = f(0)$$

$$f(0) = 1$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x}$$ ← NE POSTOJI

$$\left.\begin{array}{l} \lim_{x\to 0^+} \frac{1}{x} = \frac{1}{0^+} = +\infty \\ \lim_{x\to 0^-} \frac{1}{x} = \frac{1}{0^-} = -\infty \end{array}\right\} \neq$$

$$1.3 \quad f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}, \text{ u tački } x = 0;$$

NIJE NEPREKIDNA

$$\underbrace{\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)}_{\lim_{x\to 0} f(x)} = f(0)$$

$$f(0) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1) = -1$$

$$0 \neq 1 \neq -1$$

1.4 $f(x) = \begin{cases} (1+x)^{\frac{1}{x}}, & x \neq 0 \\ e, & x = 0 \end{cases}$, u tački $x = 0$;

$$f(0) = \lim_{x \to 0} f(x)$$

$$\left.\begin{array}{l} f(0) = e \\ \lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (1+x)^{\frac{1}{x}} = e \end{array}\right\} =$$

JESTE NEPREKIDNA

1.5 $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$, u tački $x = 2$;

~~(1)~~

1.6 $f(x) = \begin{cases} 1 - x^2, & x < 0 \\ (x-1)^2, & 0 \le x \le 2 \\ 4 - x, & x > 2 \end{cases}$, u tačkama $x = 0$ i $x = 2$.

ZA 0:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\left.\begin{aligned} &f(0) = 1 \\ &\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-1)^2 = 1 \\ &\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1-x^2) = 1 \end{aligned}\right\} =$$

$x=0$ JESTE NEPREKIDNA

ZA 2:

$$f(2) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\left.\begin{aligned} &f(2) = (2-1)^2 = 1 \\ &\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-1)^2 = 1 \\ &\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4-x) = 2 \end{aligned}\right\}$$

$1 \neq 2$

NIJE NEPREKIDNA U $x = 2$

2. Ispitati neprekidnost sledećih funkcija nad $\mathbb{R}$:

2.1 $f(x) = \begin{cases} 2x+1, & x \geq 1 \\ 3x, & x < 1 \end{cases}$;

2.2 $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x < 0 \\ 2x - 1, & x \geq 0 \end{cases}$;

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$f(0) = 2 \cdot 0 - 1 = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x - 1) = -1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin 2x}{x} \overset{\frac{0}{0}}{=}$$

$$= \lim_{x \to 0^-} \frac{\cos 2x \cdot 2}{1} = 2$$

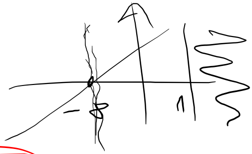
2.3 $f(x) = \begin{cases} \frac{x^2-4}{x^2+x-6}, & x > 2 \\ x+3, & x \leq 2 \end{cases}$;

2.4 $f(x) = \begin{cases} \frac{\sin 6x}{\sin 3x}, & x > 0 \\ x + 2, & x \leq 0 \end{cases}$;

2.5 $f(x) = \begin{cases} x+1, & x>0 \\ 1, & x=0 \\ \frac{\sin x}{x}, & x<0 \end{cases}$;

$$2.6\quad f(x) = \begin{cases} \frac{x-1}{\sqrt{x}-1}, & x > 1 \\ 5 - 3x, & -2 \le x \le 1 \\ \frac{6}{x-4}, & x < -2 \end{cases}.$$

3. Odrediti konstantu A tako da funkcija:



3.1 $f(x) = \begin{cases} \frac{\sqrt{1-x}-3}{8+x}, & x \neq -8 \wedge x \leq 1 \\ A, & x = -8 \end{cases}$

bude neprekidna u $x = -8$;

$-\frac{1}{6}$ $\qquad \lim_{x \to -8} f(x) = f(-8) \qquad A \qquad\qquad A = -\frac{1}{6}$

$f(-8) = A$

$$\lim_{x \to -8} f(x) = \lim_{x \to -8} \frac{\sqrt{1-x} - 3}{8+x} \;\frac{0}{0}\; \cdot \frac{\sqrt{1-x}+3}{\sqrt{1-x}+3} = \lim_{x \to -8} \frac{\overset{-1}{\cancel{1-x-9}}}{\cancel{(8+x)}(\sqrt{1-x}+3)} = -\frac{1}{6}$$

$$= \lim_{x \to -8} \frac{\frac{1}{2\sqrt{1-x}}(-1)}{1} = \frac{\frac{1}{2 \cdot 3}(-1)}{1} = -\frac{1}{6}$$

3.2 $f(x) = \begin{cases} (1+x)^{\frac{2}{x}}, & x \neq 0 \\ A, & x = 0 \end{cases}$ bude neprekidna u $x = 0$.

4. Odrediti konstantu A tako da funkcija bude neprekidna u svim tačkama domena:

4.1 $f(x) = \begin{cases} 1+x, & x \leq 1 \\ 3 - Ax^2, & x > 1 \end{cases}$;

$$3-A=2$$

$$\boxed{A=1}$$

$$\underbrace{\lim_{x\to 1^-} f(x)}_{2} = \underbrace{\lim_{x\to 1^+} f(x)}_{3-A} = \underbrace{f(1)}_{2}$$

$$f(1) = 1+1 = 2$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1+x) = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-Ax^2) = 3 - A\cdot 1^2 = 3-A$$

4.2 $f(x) = \begin{cases} (e+x)^{\sin x}, & x \geq 0 \\ \sin x + A, & x < 0 \end{cases}$;

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$f(0) = (e+0)^{\sin 0} = e^0 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\sin x + A) = 0 + A = A$$

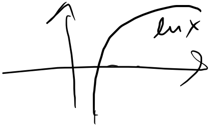
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (e+x)^{\sin x} = e^0 = 1$$

$A = 1$

4.3 $f(x) = \begin{cases} 3x - 2, & x > 1 \\ A, & x = 1 \\ x^2, & x < 1 \end{cases}$;

4.4 $f(x) = \begin{cases} 3 - 4x, & x < 0 \\ x^2 + 5x - 4A, & x \geq 0 \end{cases}$;

4.5 $f(x) = \begin{cases} (x+2)\,e^{\frac{1}{x}}, & x < 0 \\ A, & x = 0 \\ \frac{-1}{1+\ln x}, & x > 0 \end{cases}$;



$$f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$f(0) = A$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{-1}{1+\ln x} = \text{"}\frac{-1}{1+(-\infty)}\text{"} = \text{"}\frac{-1}{-\infty}\text{"} = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2)\,e^{\frac{1}{x}} = \text{"}2\,e^{\frac{1}{0^-}}\text{"} = 2 \cdot e^{-\infty}\text{"} = 2\,0$$

$$A = 0$$

4.6 $f(x) = \begin{cases} e^{\frac{1}{x}} + 3, & x < 0 \\ A, & x = 0 \\ \frac{\sin 9x}{3x}, & x > 0 \end{cases}$;

4.7 $f(x) = \begin{cases} x+1, & x>0 \\ A, & x=0 \\ \frac{x^2-3x}{x^2-2x}, & x<0 \end{cases}$.

5. Odrediti konstante A i B tako da funkcija:

5.1 $f(x) = \begin{cases} e^{-\frac{2}{x}} + 4, & x > 0 \\ A^2, & x = 0 \\ \frac{\sin Bx}{\sin 2x}, & x < 0 \end{cases}$, bude neprekidna u $x = 0$;

$$\overset{A^2}{f(0)} = \overset{4}{\lim_{x\to 0^+} f(x)} = \overset{\frac{B}{2}}{\lim_{x\to 0^-} f(x)}$$

$$f(0) = A^2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(e^{-\frac{2}{x}} + 4\right) = e^{-\frac{2}{0^+}} + 4 = e^{-\infty} + 4 = 4$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin Bx}{\sin 2x} \overset{\frac{0}{0}}{=} \lim_{x\to 0^+} \frac{\overset{1}{\cancel{\cos Bx}} \cdot B}{\cancel{\cos 2x} \cdot 2} = \frac{B}{2}$$

$$\lim_{x\to 0} \frac{\overset{1}{\cancel{\frac{\sin Bx}{Bx}}} \cdot B\cancel{x}}{\underset{1}{\cancel{\frac{\sin 2x}{2x}}} \cdot 2\cancel{x}}$$

$$A^2 = 4 = \frac{B}{2}$$

$$A = \pm 2 \qquad B = 8$$

5.2 $f(x) = \begin{cases} -\sin x, & x \le -\frac{\pi}{2} \\ A\sin x + B, & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ \cos x & x \ge \frac{\pi}{2} \end{cases}$

$-A+B$ bude neprekidna nad $\mathbb{R}$;

$-A+B=1$
$A+B=0$

$2B=1$

$\boxed{B=\frac{1}{2} \quad A=-\frac{1}{2}}$

$x=-\frac{\pi}{2}$

$$\lim_{x\to-\frac{\pi}{2}^+} f(x) = \lim_{x\to-\frac{\pi}{2}^-} f(x) = f\left(-\frac{\pi}{2}\right) \quad 1$$

$$f\left(-\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$\lim_{x\to-\frac{\pi}{2}^+} f(x) = \lim_{x\to-\frac{\pi}{2}^+} (A\sin x + B) = A\sin\left(-\frac{\pi}{2}\right) + B = -A\cdot 1 + B = -A+B$$

$$\lim_{x\to-\frac{\pi}{2}^-} f(x) = \lim_{x\to-\frac{\pi}{2}^-} (-\sin x) = -\sin\left(-\frac{\pi}{2}\right) = 1$$

$x=\frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$$

$A+B$ 0 0

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+} \cos x = \cos\frac{\pi}{2} = 0$$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} (A\sin x + B) = A\sin\frac{\pi}{2} + B = A+B$$

5.3 $f(x) = \begin{cases} (\cos x)^{\frac{1}{\sin^2 x}}, & x > 0 \\ A, & x = 0 \\ B(1 + e^{\frac{1}{x}}) & x < 0 \end{cases}$, bude neprekidna u $x = 0$.

$$\overset{e^{-\frac{1}{2}}}{\lim_{x\to 0^+} f(x)} = \overset{B}{\lim_{x\to 0^-} f(x)} = \overset{A}{f(0)} \qquad A = B = e^{-\frac{1}{2}}$$

$$f(0) = A$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} B(1+e^{\frac{1}{x}}) = B(1+e^{\frac{1}{0^-}}) = B(1+0) = B$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (\cos x)^{\frac{1}{\sin^2 x}} \overset{1^\infty}{=} \lim_{x\to 0^+} (1+(\cos x - 1))^{\frac{1}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} (1+(\cos x-1))^{\frac{1}{\cos x - 1}\cdot(\cos x - 1)\cdot\frac{1}{\sin^2 x}}$$

$$= e^{\lim_{x\to 0^+} \frac{\cos x - 1}{\sin^2 x}} \overset{\frac{0}{0}}{=} e^{\lim_{x\to 0^+} \frac{-(1-\cos x)}{1-\cos^2 x}} = e^{\lim_{x\to 0^+} \frac{-\cancel{(1-\cos x)}}{\cancel{(1-\cos x)}(1+\cos x)}} = e^{-\frac{1}{2}}$$

$$e^{\lim_{x\to 0^+} \frac{-\cancel{\sin x}}{2\cancel{\sin x}\cdot\cos x}} = e^{-\frac{1}{2}}$$

$1^\infty, 0^0, \infty^0$

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{\sin^2 x}} \overset{1^\infty}{=} M \quad / \ln$$

$$\ln M = \ln \lim_{x\to 0^+} (\cos x)^{\frac{1}{\sin^2 x}} = \lim_{x\to 0^+} \frac{1}{\sin^2 x} \ln(\cos x)$$

$$= \lim_{x\to 0^+} \left(\frac{1}{\sin^2 x} \cdot \ln(\cos x) \right) = \lim_{x\to 0^+} \frac{\ln(\cos x)}{\sin^2 x} \overset{\frac{0}{0}}{=}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{2\sin x \cdot \cos x} = \lim_{x\to 0^+} \frac{-\cancel{\sin x}}{2\cancel{\sin x}\cos^2 x} = -\frac{1}{2}$$

$$\ln M = -\frac{1}{2}$$

$$M = e^{-\frac{1}{2}}$$

1^∞

$(1+\cancel{0})^{\frac{1}{0}}$

$$\lim_{x\to 0}(\sin x)^x \overset{0^0}{=} A \quad / \ln$$

$$\ln A = \ln \lim_{x\to 0}(\sin x)^x = \lim_{x\to 0} \ln(\sin x)^{x}$$

$$= \lim_{x\to 0} x\ln(\sin x) \overset{0\cdot\infty}{=} \lim_{x\to 0} \frac{\ln(\sin x)}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{=}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{-x^2\cos x}{\sin x} \overset{\frac{0}{0}}{=}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\cdot(-x\cos x)\right) = 1\cdot(-0\cdot 1) = 0$$

$$\ln A = 0 \Rightarrow A = e^0 = 1$$