INTEGRALI IRACIONALNIH FUNKCIJA

$$\int \frac{p_n(x)}{\sqrt{ax^2+bx+c}}\,dx$$

I $p_n(x)=1$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+c$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right|+c$$

① $$\int \frac{1}{\sqrt{x^2+2x+3}}\,dx = \int \frac{1}{\sqrt{(x+1)^2+2}}\,dx = *$$

$$x^2+\underline{2x}+3 = \underbrace{x^2+2\cdot x\,\boxed{1}+1}-1+3=(x+1)^2+2$$

$$(A+B)^2=A^2+2AB+B^2$$

smena: $x+1=t$
$dx=dt$

$$* \int \frac{1}{\sqrt{t^2+2}}\,dt = \ln\left|t+\sqrt{t^2+2}\right|+c$$

$$= \ln\left|x+1+\sqrt{(x+1)^2+2}\right|+C$$

$$= \ln\left|x+1+\sqrt{x^2+2x+3}\right|+C$$

(2) $$\int \frac{1}{\sqrt{4-3x-x^2}}\,dx = \int \frac{1}{\sqrt{\frac{25}{4}-\left(x+\frac{3}{2}\right)^2}}\,dx = \int \frac{1}{\sqrt{\left(\frac{5}{2}\right)^2-t^2}}\,dt = \arcsin \frac{t}{\frac{5}{2}} + C$$

$$4-3x-x^2 = -\left(x^2+\boxed{3}x-4\right)$$

$$= -\left(\underbrace{x^2+2x\cdot\boxed{\tfrac{3}{2}}+\tfrac{9}{4}}-\tfrac{9}{4}-4\right)$$

$$= -\left(\left(x+\tfrac{3}{2}\right)^2-\tfrac{25}{4}\right)$$

$$= \frac{25}{4}-\left(x+\frac{3}{2}\right)^2$$

Смена:
$x+\frac{3}{2}=t$
$dx=dt$

$$= \arcsin \frac{2}{5}\left(x+\frac{3}{2}\right)+C$$

(3) $$\int \frac{1}{\sqrt{4x-2x^2}}\,dx = \int \frac{1}{\sqrt{2\left(1-(x-1)^2\right)}}\,dx = \int \frac{1}{\sqrt{2}\,\sqrt{1-(x-1)^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{1^2-t^2}}\,dt$$

$$-2x^2+4x = -2\left(x^2-2x\right)$$

$$= -2\left(\underbrace{x^2-2x\,\boxed{1}+1}-1\right)$$

$$= -2\left((x-1)^2-1\right)$$

$$= 2\left(1-(x-1)^2\right)$$

Смена: $x-1=t$
$dx=dt$

$$= \frac{1}{\sqrt{2}}\arcsin t + C$$

$$= \frac{1}{\sqrt{2}}\arcsin(x-1)+C$$

II $P_1(x) = Ax + B$

MOŽE I KAO III TIP, NE MORA POSEBNO DA SE UČI

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$$

(1)

$$\int \frac{x+5}{\sqrt{x^2+10x+3}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = \frac{1}{\cancel{2}}\,\frac{t^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} + C = \sqrt{x^2+10x+3} + C$$

Smena:

$$x^2+10x+3 = t$$
$$(2x+10)\,dx = dt$$
$$2(x+5)\,dx = dt$$
$$(x+5)\,dx = \frac{1}{2}\,dt$$

(2)

$$\int \frac{-2x+5}{\sqrt{-x^2+4x-3}}\,dx = \int \frac{-2x+4+1}{\sqrt{-x^2+4x-3}}\,dx$$

Smena:

$$-x^2+4x-3 = t$$
$$(-2x+4)\,dx = dt$$

$$= \int \frac{-2x+4}{\sqrt{-x^2+4x-3}}\,dx + \int \frac{1}{\sqrt{-x^2+4x-3}}\,dx$$

$$= \int \frac{1}{\sqrt{t}}\,dt + \int \frac{1}{\sqrt{1-(x-2)^2}}\,dx = \int \frac{dt}{\sqrt{t}} + \int \frac{du}{\sqrt{1-u^2}}$$

$$-x^2+4x-3 = -(x^2-4x+3)$$
$$= -(\underbrace{x^2-2x\boxed{2}+4}-4+3)$$
$$= -((x-2)^2-1)$$
$$= 1-(x-2)^2$$

Smene

$$x-2 = u$$
$$dx = du$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + \arcsin u + C$$
$$= 2\sqrt{-x^2+4x-3} + \arcsin(x-2) + C$$

II

$n \geq 1$

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = Q_{n-1}'(x)\sqrt{ax^2+bx+c} + Q_{n-1}(x)\left(\sqrt{ax^2+bx+c}\right)' + \lambda \frac{1}{\sqrt{ax^2+bx+c}}$$

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = Q_{n-1}'(x)\sqrt{ax^2+bx+c} + Q_{n-1}(x)\frac{1}{2\sqrt{ax^2+bx+c}}(ax^2+bx+c)' + \lambda \frac{1}{\sqrt{ax^2+bx+c}}$$

$$\frac{P_n(x)}{\cancel{\sqrt{ax^2+bx+c}}} = Q_{n-1}'(x)\sqrt{ax^2+bx+c} + Q_{n-1}(x)\frac{1}{2\cancel{\sqrt{ax^2+bx+c}}}(2ax+b) + \lambda \frac{1}{\cancel{\sqrt{ax^2+bx+c}}}$$

$\cdot \sqrt{ax^2+bx+c}$

$$P_n(x) = Q_{n-1}'(x)(ax^2+bx+c) + Q_{n-1}(x)\frac{1}{2}(2ax+b) + \lambda$$

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$$\int \frac{x^2+2}{\sqrt{x^2+4x+5}}\,dx = (Ax+B)\sqrt{x^2+4x+5} + \lambda \int \frac{dx}{\sqrt{x^2+4x+5}}$$

$$\frac{x^2+2}{\sqrt{x^2+4x+5}} = (Ax+B)'\sqrt{x^2+4x+5} + (Ax+B)\left(\sqrt{x^2+4x+5}\right)' + \lambda \frac{1}{\sqrt{x^2+4x+5}}$$

$$\frac{x^2+2}{\cancel{\sqrt{x^2+4x+5}}} = A\sqrt{x^2+4x+5} + (Ax+B)\cdot \frac{1}{2\cancel{\sqrt{x^2+4x+5}}}(2x+4) + \lambda \frac{1}{\cancel{\sqrt{x^2+4x+5}}}$$

$\sqrt{x^2+4x+5}$

$$x^2+2 = A(x^2+4x+5) + (Ax+B)\frac{1}{2}(2x+4) + \lambda$$

$$x^2+2 = Ax^2+4Ax+5A + Ax^2+2Ax+Bx+2B+\lambda$$

$x+2$

$$x^2+2 = 2Ax^2+(6A+B)x + 5A+2B+\lambda$$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$6A+B = 0 \Rightarrow B = -3$

$5A+2B+\lambda = 2$

$\frac{5}{2} - 6 + \lambda = 2$

$\lambda = 8 - \frac{5}{2}$

$\lambda = \frac{11}{2}$

$$\int \frac{x^2+2}{\sqrt{x^2+4x+5}}\,dx = \left(\frac{1}{2}x-3\right)\sqrt{x^2+4x+5} + \frac{11}{2}\boxed{\int \frac{1}{\sqrt{x^2+4x+5}}\,dx}$$

$$\int \frac{1}{\sqrt{x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+1}}\,dx = \int \frac{1}{\sqrt{t^2+1}}\,dt = \ln\left|t+\sqrt{t^2+1}\right|+c$$

$$= \ln\left|x+2+\sqrt{(x+2)^2+1}\right|+c$$

$$x^2+4x+5 = x^2+2x\,\boxed{2}+4-4+5$$

$$= (x+2)^2+1$$

Smeno

$x+2=t$

$dx=dt$

② $\int \frac{-2x+5}{\sqrt{-x^2+4x+3}}\,dx = A\cdot\sqrt{-x^2+4x+3} + \lambda \int \frac{1}{\sqrt{-x^2+4x+3}}\,dx \quad /'$

$(c\,f(x))' = c\,f'(x)$

$$\frac{-2x+5}{\sqrt{-x^2+4x+3}} = A\,\frac{1}{2\sqrt{-x^2+4x+3}}\cdot(-2x+4) + \lambda\,\frac{1}{\sqrt{-x^2+4x+3}} \quad / \sqrt{-x^2+4x+3}$$

$$-2x+5 = A\cdot\frac{1}{2}\cdot(-2x+4) + \lambda$$

$$-2x+5 = -Ax + 2A + \lambda$$

$$-A = -2 \Rightarrow A = 2$$

$$2A + \lambda = 5 \qquad \lambda = 1$$

$$-x^2+4x+3 = -(x^2-4x-3)$$

$$= -(x^2 - 2\cdot x\cdot \boxed{2} + 4 - 4 - 3)$$

$$= -((x-2)^2 - 7)$$

$$= 7 - (x-2)^2$$

$$\int \frac{-2x+5}{\sqrt{-x^2+4x+3}}\,dx = 2\sqrt{-x^2+4x+3} + \int \frac{1}{\sqrt{-x^2+4x+3}}\,dx$$

$$\int \frac{1}{\sqrt{-x^2+4x+3}}\,dx = \int \frac{1}{\sqrt{7-(x-2)^2}}\,dx$$

$x-2 = t$

$dx = dt$

$$= \int \frac{1}{\sqrt{(\sqrt{7})^2 - t^2}}\,dt$$

$$= \arcsin\frac{t}{\sqrt{7}} + C$$

$$= \arcsin\frac{x-2}{\sqrt{7}} + C$$