Vektori

December 21, 2021

1. Neka su vektori $\vec{a}$ i $\vec{b}$ takvi da je $|\vec{a}| = 2$, $\left|\vec{b}\right| = 1$ i $\sphericalangle\left(\vec{a}, \vec{b}\right) = \dfrac{\pi}{3}$.

Izračunati:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\sphericalangle(\vec{a}, \vec{b}) = 2 \cdot 1 \cdot \cos\frac{\pi}{3} = \cancel{2} \cdot \frac{1}{\cancel{2}} = 1$$

$$\begin{aligned}\left(\vec{a} + 2\vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right) &= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{a} - 2\vec{b}\,\vec{b} \\ &= |\vec{a}|^2 - \vec{a}\,\vec{b} + 2\vec{a}\cdot\vec{b} - 2|\vec{b}|^2 \\ &= 4 + 1 - 2 \\ &= 3\end{aligned}$$

$$\begin{aligned}\left(\vec{a} - \vec{b}\right) \cdot \left(\vec{a} + \vec{b}\right) &= \vec{a}\cdot\vec{a} + \cancel{\vec{a}\,\vec{b}} - \cancel{\vec{b}\,\vec{a}} - \vec{b}\cdot\vec{b} \\ &= |\vec{a}|^2 - |\vec{b}|^2 \\ &= 4 - 1 \\ &= 3\end{aligned}$$

$$\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\underbrace{\cos\sphericalangle(\vec{a},\vec{a})}_{1}$$

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$

$$(A-B)(A+B) = A^2 - B^2$$

$$\cancel{\vec{a}^2 - \vec{b}^2}$$

$$\begin{aligned}\alpha(\vec{a}\cdot\vec{b}) &= (\alpha\vec{a})\,\vec{b} \\ &= \vec{a}\cdot(\alpha\vec{b})\end{aligned}$$

2. Naći intenzitet vektora $\vec{a} = \vec{p} - 2\vec{q}$, ako je $|\vec{p}| = 2$, $|\vec{q}| = \sqrt{3}$ i $\sphericalangle(\vec{p}, \vec{q}) = \dfrac{\pi}{6}$.

$$|\vec{a}|^2 = \vec{a} \cdot \vec{a} \Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

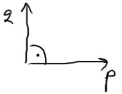
$$\begin{aligned}
\underline{\vec{a} \cdot \vec{a}} &= (\vec{p} - 2\vec{q}) \cdot (\vec{p} - 2\vec{q}) \\
&= \vec{p} \cdot \vec{p} - 2\vec{p} \cdot \vec{q} - 2\vec{q} \cdot \vec{p} + 4\vec{q} \cdot \vec{q} \\
&= |\vec{p}|^2 - 4\underline{\vec{p} \cdot \vec{q}} + 4|\vec{q}|^2 \\
&= 4 - 4 \cdot |\vec{p}| \cdot |\vec{q}| \cdot \cos\sphericalangle(\vec{p}, \vec{q}) + 4 \cdot 3 \\
&= 16 - 4 \cdot 2 \cdot \sqrt{3} \cos \frac{\pi}{6} \\
&= 16 - \cancel{8}^{4}\sqrt{3}\, \frac{\sqrt{3}}{\cancel{2}} \\
&= \underline{4}
\end{aligned}$$

$$|\vec{a}| = \sqrt{4} = 2$$

3. Odrediti realan parametar α tako da vektori $\vec{p} = \alpha\vec{a} + 17\vec{b}$ i $\vec{q} = 3\vec{a} - \vec{b}$ budu uzajamno normalni, ako je $|\vec{a}| = 2$, $\left|\vec{b}\right| = 5$ i $\sphericalangle\left(\vec{a}, \vec{b}\right) = \frac{2\pi}{3}$.

$\perp$ = NORMALNI = ORTOGONALNI

$\vec{p} \perp \vec{q}$



$\cos\sphericalangle(\vec{p},\vec{q}) = \cos\frac{\pi}{2} = 0$

$\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}|\cos\sphericalangle(\vec{p},\vec{q})$

$\Rightarrow \vec{p}\cdot\vec{q} = 0$

$$\begin{array}{r} 17\cdot 25 \\ \hline 85 \\ 34 \\ \hline 425 \end{array}$$

$[0,\pi]$

$\vec{p} \perp \vec{q} \iff \vec{p}\cdot\vec{q} = 0$

$$\begin{aligned}
\vec{p}\cdot\vec{q} &= (\alpha\vec{a} + 17\vec{b})\cdot(3\vec{a} - \vec{b}) \\
&= 3\alpha\,\vec{a}\cdot\vec{a} - \alpha\,\vec{a}\cdot\vec{b} + 51\,\vec{b}\cdot\vec{a} - 17\,\vec{b}\cdot\vec{b} \\
&= 3\alpha|\vec{a}|^2 + (51-\alpha)\,\vec{a}\cdot\vec{b} - 17\cdot|\vec{b}|^2 \\
&= 3\alpha\cdot 4 + (51-\alpha)|\vec{a}||\vec{b}|\cdot\cos\sphericalangle(\vec{a},\vec{b}) - 17\cdot 25 \\
&= 12\alpha + (51-\alpha)\cdot 2\cdot 5\cdot\cos\frac{2\pi}{3} - 425 \\
&= 12\alpha + (51-\alpha)\,\cancel{10}^{5}\cdot\left(-\frac{1}{\cancel{2}}\right) - 425 \\
&= 12\alpha - 255 + 5\alpha - 425 \\
&= 17\alpha - 680
\end{aligned}$$

$$\begin{aligned}
\vec{p}\cdot\vec{q} &= 17\alpha - 680 \\
0 &= 17\alpha - 680 \\
17\alpha &= 680 \\
&\boxed{\alpha = 40}
\end{aligned}$$

4. Koji ugao obrazuju jedinični vektori $\vec{a}$ i $\vec{b}$ ako su vektori $\vec{p} = \vec{a} + 2\vec{b}$ i $\vec{q} = 5\vec{a} - 4\vec{b}$ uzajamno normalni?

$$|\vec{a}| = |\vec{b}| = 1$$

$$\vec{p} \perp \vec{q} \iff \vec{p} \cdot \vec{q} = 0$$

$$\vec{p} \cdot \vec{q} = 0$$

$$(\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 4\vec{b}) = 0$$

$$5\vec{a}\cdot\vec{a} - 4\vec{a}\cdot\vec{b} + 10\,\vec{b}\cdot\vec{a} - 8\vec{b}\cdot\vec{b} = 0$$

$$5|\vec{a}|^2 + 6\,\underline{\vec{a}\cdot\vec{b}} - 8|\vec{b}|^2 = 0$$

$$5 + 6|\vec{a}||\vec{b}|\cos\measuredangle(\vec{a},\vec{b}) - 8 = 0$$

$$-3 + 6\cdot 1\cdot 1\cdot \cos\measuredangle(\vec{a},\vec{b}) = 0$$

$$6\cos\measuredangle(\vec{a},\vec{b}) = 3$$

$$\cos\measuredangle(\vec{a},\vec{b}) = \frac{1}{2}$$

$$\underline{\underline{\measuredangle(\vec{a},\vec{b}) = \frac{\pi}{3}}}$$

Напомена:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\measuredangle(\vec{a},\vec{b})$$

$$= 1\cdot 1\cdot \frac{1}{2}$$

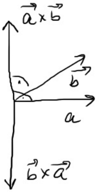
$$= \frac{1}{2}$$

5. Izračunati površinu paralelograma konstruisanog nad vektorima $\vec{p} = 2\vec{b} - \vec{a}$ i $\vec{q} = 3\vec{a} + 2\vec{b}$ ako je $|\vec{a}| = \left|\vec{b}\right| = 5$ i $\sphericalangle\left(\vec{a}, \vec{b}\right) = \frac{\pi}{4}$.

$$P_{\square} = |\vec{p} \times \vec{q}|$$

$$\begin{aligned}\vec{p} \times \vec{q} &= (2\vec{b} - \vec{a}) \times (3\vec{a} + 2\vec{b}) \\ &= 6\,\vec{b} \times \vec{a} + 4\underbrace{\vec{b} \times \vec{b}}_{0} - 3\underbrace{\vec{a} \times \vec{a}}_{0} - 2\underbrace{\vec{a} \times \vec{b}}_{-\vec{b} \times \vec{a}} \\ &= 6\,\vec{b} \times \vec{a} + 2\,\vec{b} \times \vec{a} \\ &= 8\,\vec{b} \times \vec{a}\end{aligned}$$

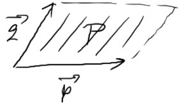
$$\begin{aligned}P_{\square} &= |\vec{p} \times \vec{q}| = |8\,\vec{b} \times \vec{a}| = |8|\,|\vec{b} \times \vec{a}| \\ &= 8\,|\vec{b}|\,|\vec{a}| \sin\sphericalangle(\vec{a}, \vec{b}) = 8 \cdot 5 \cdot 5 \sin\sphericalangle\frac{\pi}{4} \\ &= 200 \cdot \frac{\sqrt{2}}{2} = 100\sqrt{2}\end{aligned}$$



$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\alpha \vec{a}$$

$$|\alpha \vec{a}| = |\alpha|\,|\vec{a}|$$



$$P = |\vec{p} \times \vec{q}|$$

$$\begin{aligned}\alpha(\vec{a} \times \vec{b}) &= (\alpha \vec{a}) \times \vec{b} \\ &= \vec{a} \times (\alpha \vec{b})\end{aligned}$$

$\vec{a} \times \vec{a}$

$\Longrightarrow$ $\vec{a} \times \vec{b}$:

1) $\vec{a} \times \vec{b} \perp \vec{a} \wedge \vec{a} \times \vec{b} \perp \vec{b}$

2) $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \cdot \sin\sphericalangle(\vec{a}, \vec{b})$

3) pravilo desnog triedr

$$\Rightarrow \vec{a} \times \vec{a} = 0$$

6. Za koje vrednosti realnog parametra k će vektori $\vec{p} = k\vec{a} + 5\vec{b}$ i $\vec{q} = 3\vec{a} - \vec{b}$ biti kolinearni ako vektori $\vec{a}$ i $\vec{b}$ nisu kolinearni?

$$\vec{p} \parallel \vec{q} \iff \vec{p} \times \vec{q} = 0$$

$$\vec{p} \times \vec{q} = 0$$

$$(k\vec{a} + 5\vec{b}) \times (3\vec{a} - \vec{b}) = 0$$

$$3k\,\cancel{\vec{a}\times\vec{a}}^{\,0} - k\vec{a}\times\vec{b} + 15\vec{b}\times\vec{a} - 5\,\cancel{\vec{b}\times\vec{b}}^{\,0} = 0$$

$$-k\vec{a}\times\vec{b} - 15\vec{a}\times\vec{b} = 0$$

$$(-k-15)\,\vec{a}\times\vec{b} = 0$$

$$-k-15 = 0 \quad \vee \quad \underbrace{\vec{a}\times\vec{b} = 0}$$

$$\boxed{k = -15}$$

Ово не може никад да се деси јер $\vec{a}$ и $\vec{b}$ нису колинеарни па је $\vec{a}\times\vec{b} \neq 0$ увек.

$\vec{p}$ i $\vec{q}$ kolinearni $\iff$

$\vec{p} \parallel \vec{q}$ (ili $\vec{p} \equiv \vec{q}$)

$\iff \vec{q} = \alpha\vec{p}$, $\alpha \in \mathbb{R}$

$\iff \vec{p} \times \vec{q} = 0$