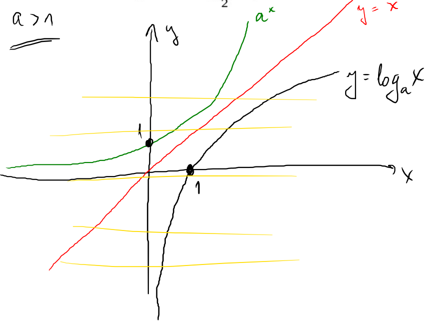
Logaritamska funkcija, jednačine i nejednačine

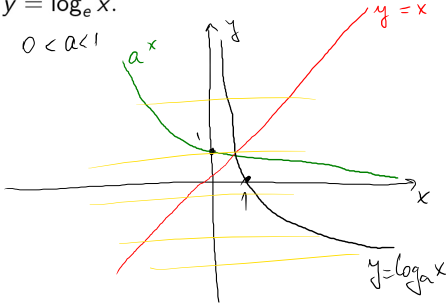
16. септембар 2024.

Logaritamska funkcija

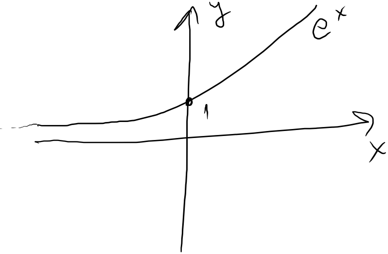
$$y = \log_a x, \;\; a > 0, \;\; a \neq 1,$$

definisana je za sve $x > 0$. Presečna tačka sa x-osom je $(1, 0)$. Logaritamska funkcija je opadajuća za osnovu $0 < a < 1$, npr. $y = \log_{\frac{1}{2}} x$, a rastuća za $a > 1$, npr. $y = \log_e x$.



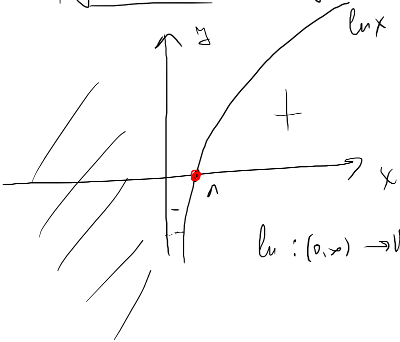


$\boxed{y = e^x}$



$e^x : \mathbb{R} \to (0, \infty)$

$\boxed{y = \ln x} = \log_e x$



$\ln : (0, \infty) \to \mathbb{R}$

Logaritamska jednačina je oblika

$$\log_a x = b, \quad a, \cancel{b} > 0, \quad a \neq 1$$

Logaritamska i eksponencijalna funkcija sa istom osnovom su međusobno inverzne funkcije, pa imamo

$$\boxed{\log_a x = b \quad \Leftrightarrow \quad x = a^b.}$$

$\log_a x = b \iff x = a^b$

$\boxed{\ln x = b \iff x = e^b}$

Kako je logaritamska funkcija injektivna, važi

$$\log_a x = \log_a y \quad \Leftrightarrow \quad x = y.$$

$f(x) = f(y) \Rightarrow x = y$

$\cancel{\log}_a x = \cancel{\log}_a y$!

Osobine logaritama:

$x, y > 0,\ \ a, b, c \in \mathbb{R}^+ \setminus \{1\}$

1) $\log_a(x \cdot y) = \log_a x + \log_a y$

2) $\log_a \frac{x}{y} = \log_a x - \log_a y$

3) $\log_a x^n = n \cdot \log_a x$

4) $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$

5) $\log_a 1 = 0 \ \wedge \ \log_a a = 1$

6) $\log_a b = \frac{\log_c b}{\log_c a}$

7) $\log_a b = \frac{1}{\log_b a}$

8) $\log_{a^r} x = \frac{1}{r} \log_a x$

9) $a^{y \log_a x} = \left(a^{\log_a x}\right)^y = x^y$

$\ln 1 = 0$ $\qquad$ $\log_a a = \_$

$e^0 = 1$ $\qquad$ $a^1 = a$

$\boxed{\ln x = b \Leftrightarrow e^b = x}$

Primer: Rešiti jednacine:

услове

$x-1>0 \quad \wedge \quad x+2\geq 0$

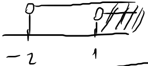
$x>1 \qquad x\geq -2$

1. $\log(x-1)+2\log\sqrt{x+2}=1$

$$\log(x-1)+2\log(x+2)^{\frac{1}{2}}=1$$

$$\log(x-1)+\log(x+2)=1$$

$$\log_{10}\big((x-1)(x+2)\big)=1$$



$x\in(1,\infty)$

$\wedge \quad \sqrt{x+2}>0$

$x+2\neq 0$

$x\neq -2$

$$\boxed{\log_a x=b \iff x=a^b}$$

$$(x-1)(x+2)=10$$

$$x^2+x-12=0$$

$$x_{1,2}=\frac{-1\pm\sqrt{1+48}}{2}=\frac{-1\pm 7}{2}=\begin{cases}\cancel{-4}\\ 3\end{cases}$$

2. $\log_2^2 x + 2\log_2 \sqrt{x} = 2$

$$\log_a^2 x = \log_a x \cdot \log_a x = (\log_a x)^2$$

$$(\log_2 x)^2 + \log_2 x = 2$$

условие

$x > 0$

$$\boxed{D = (0, \infty)}$$

$\log_2 x = t$

$$t^2 + t - 2 = 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \quad 1; \ -2$$

$t = 1$

$\log_2 x = 1$

$$\boxed{x = 2}$$

$t = -2$

$\log_2 x = -2$

$$\boxed{x = 2^{-2} = \frac{1}{4}}$$

3. $\log_4(x-2) + \log_{16}(x-2) + \log_2(x-2) = 7$

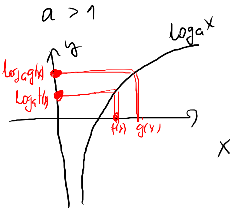
Logaritamske nejednačine:

- ako je $a > 1$, onda važi:

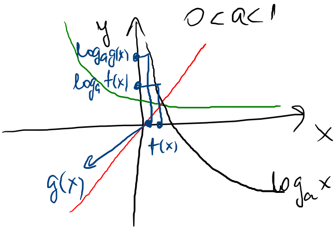
$$\log_a f(x) \leq \log_a g(x) \Leftrightarrow f(x) \leq g(x)$$

- ako je $0 < a < 1$, onda važi:

$$\log_a f(x) \leq \log_a g(x) \Leftrightarrow f(x) \geq g(x).$$



$0 < a < 1$



Primer: Resiti nejednacine:

1. $\log_{\frac{1}{4}}(6+2x) > 0$

услов

$\frac{1}{4} < 1$

$$6+2x < \left(\frac{1}{4}\right)^0$$

$$6+2x > 0$$

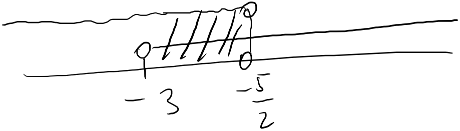
$$6+2x < 1$$

$$2x > -6$$

$$x > -3$$

$$2x < -5$$

$$x < -\frac{5}{2}$$



$$\boxed{x \in \left(-3, -\frac{5}{2}\right)}$$

$$\log_{4^{-1}}(6+2x) > 0$$

$$-\log_4(6+2x) > 0 \quad / (-1)$$

$$\log_4(6+2x) < 0$$

2. $\log_{\frac{1}{9}}(x^2 - 4) \geq \log_{\frac{1}{9}}(2x - 1)$

Условия

$\log_4 x > 0$

$\wedge \log_2 x > 0$

$\wedge x > 0$

$x > 1 \wedge x > 0$

$\boxed{x > 1}$

$\boxed{D = (1, \infty)}$

3. $\log_2(\log_4 x) + \log_4(\log_2 x) \le -4$

$\log_2(\log_4 x) + \log_{2^2}(\log_2 x) \le -4$

$\log_2(\log_4 x) + \frac{1}{2}\log_2(\log_2 x) \le -4$

$\log_2(\log_4 x) + \log_2(\log_2 x)^{\frac{1}{2}} \le -4$

$\log_2\left(\log_4 x \cdot (\log_2 x)^{\frac{1}{2}}\right) \le -4$

$\log_4 x \cdot (\log_2 x)^{\frac{1}{2}} \le 2^{-4}$

$\frac{1}{2}\log_2 x \cdot (\log_2 x)^{\frac{1}{2}} \le \frac{1}{16} \Big/ \cdot 16$

$8 \cdot \log_2 x \cdot \sqrt{\log_2 x} \le 1$

$\sqrt{\log_2 x} = t$ $\quad 8 t^2 \cdot t \le 1$

$8t^3 \le 1$

$(8t^3 - 1) \le 0$ $\quad t^3 \le \frac{1}{8}$

$(2t-1)(4t^2+2t+1) \le 0$ $\quad t^3 \le \left(\frac{1}{2}\right)^3$

$2t - 1 \le 0$ $\quad t \le \frac{1}{2}$

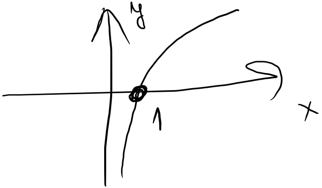
$t \le \frac{1}{2}$

$\sqrt{\log_2 x} \le \frac{1}{2}$

$\log_2 x \le \frac{1}{4}$

$\boxed{x \le 2^{\frac{1}{4}}} = \sqrt[4]{2}$

$y = \ln x$



$x > 0$

$\ln : (0, \infty) \to \mathbb{R}$

$\ln A = 3$

$A = e^3$

$\ln 1 = 0$

$$\boxed{\ln A = B \iff A = e^B}$$