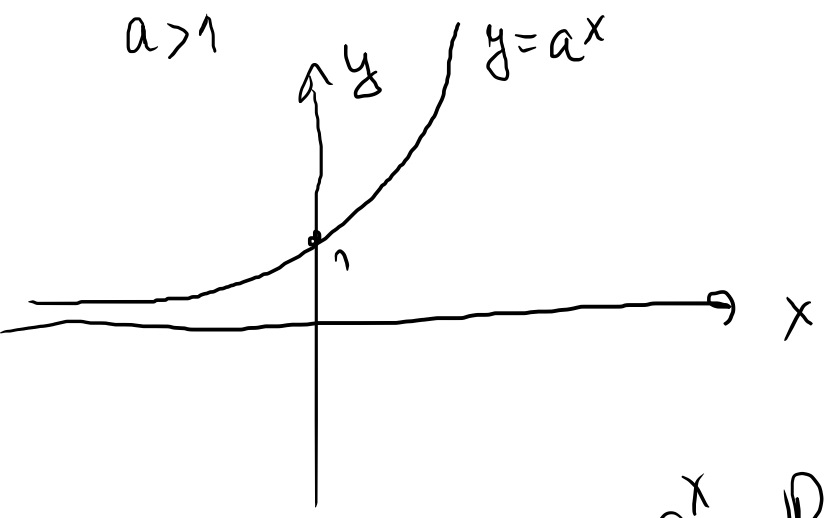
Eksponencijlna funkcija, jednačine i nejednačine

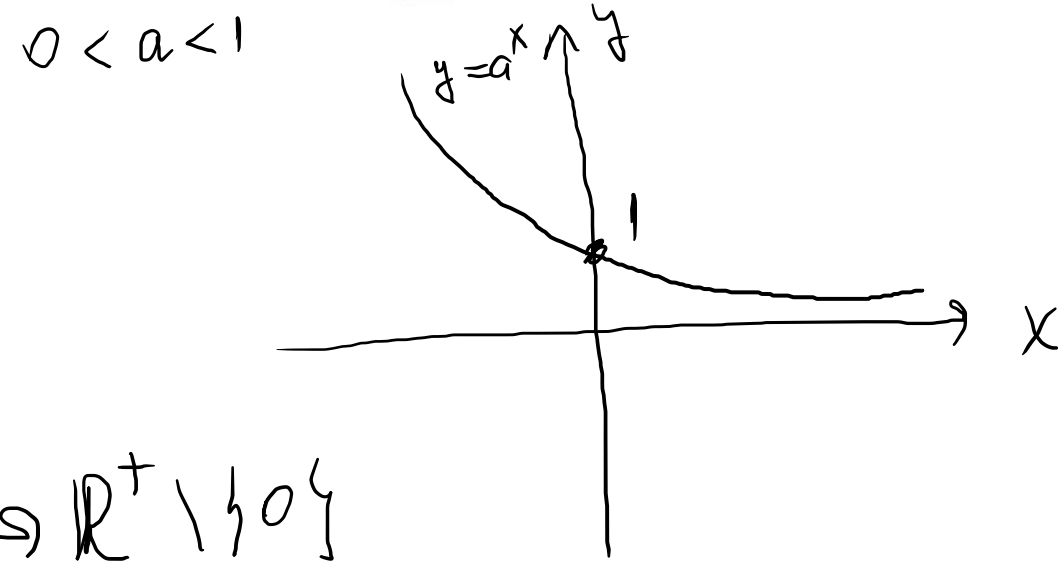
16. септембар 2024.

Eksponencijalna funkcija

$$y = a^x, \ a > 0, \ a \neq 1,$$

definisana je za svako $x \in \mathbb{R}$ i pozitivna je na celom domenu.
Seče y-osu u tački $(0, 1)$.
Eksponencijalna funkcija je rastuća za $a > 1$, npr. $y = e^x$, a opadajuća za $0 < a < 1$, npr. $y = e^{-x} = \left(\frac{1}{e}\right)^x$.



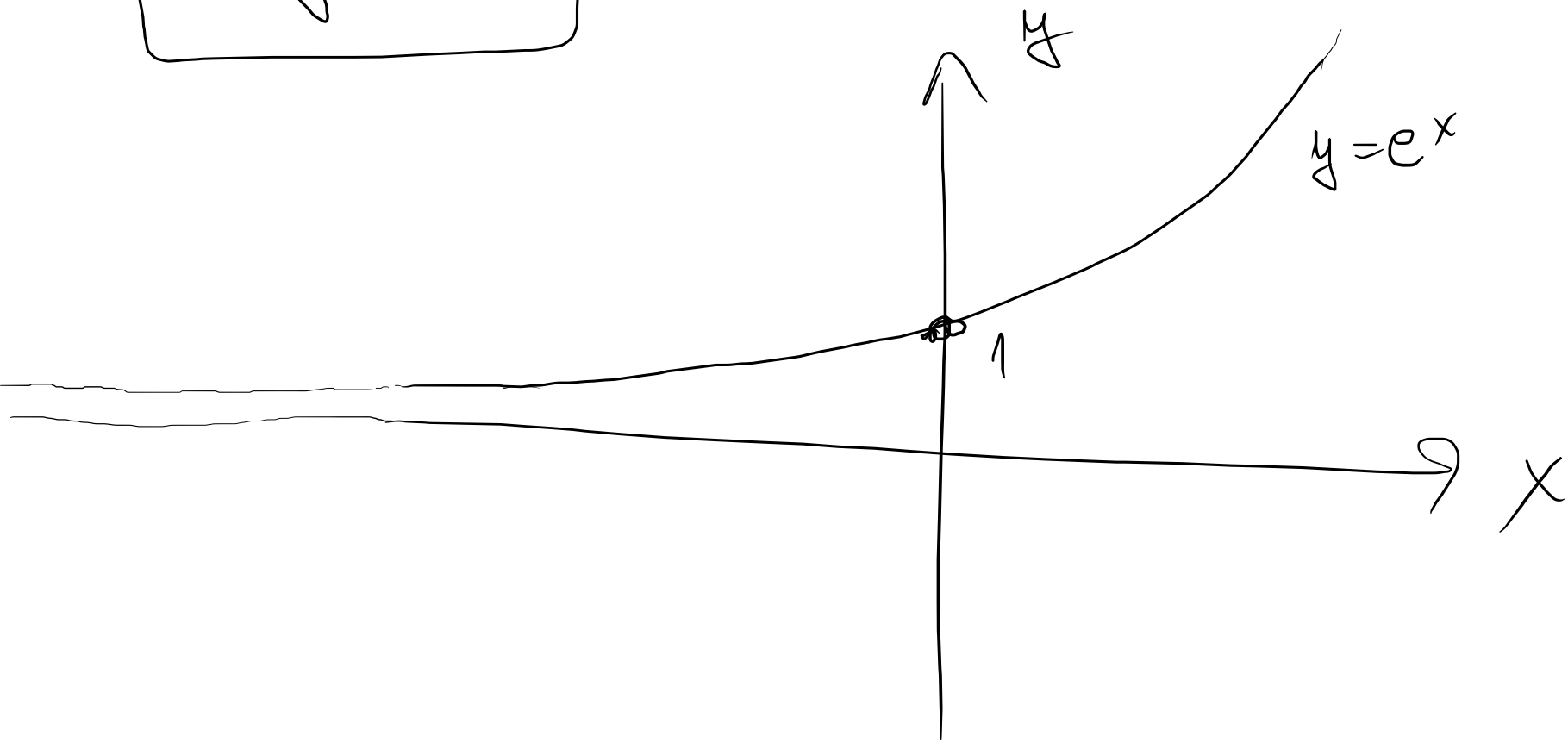


$$a^x : \mathbb{R} \to \mathbb{R}^+ \setminus \{0\}$$

$$y = e^x$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to +\infty} e^x = \infty$$

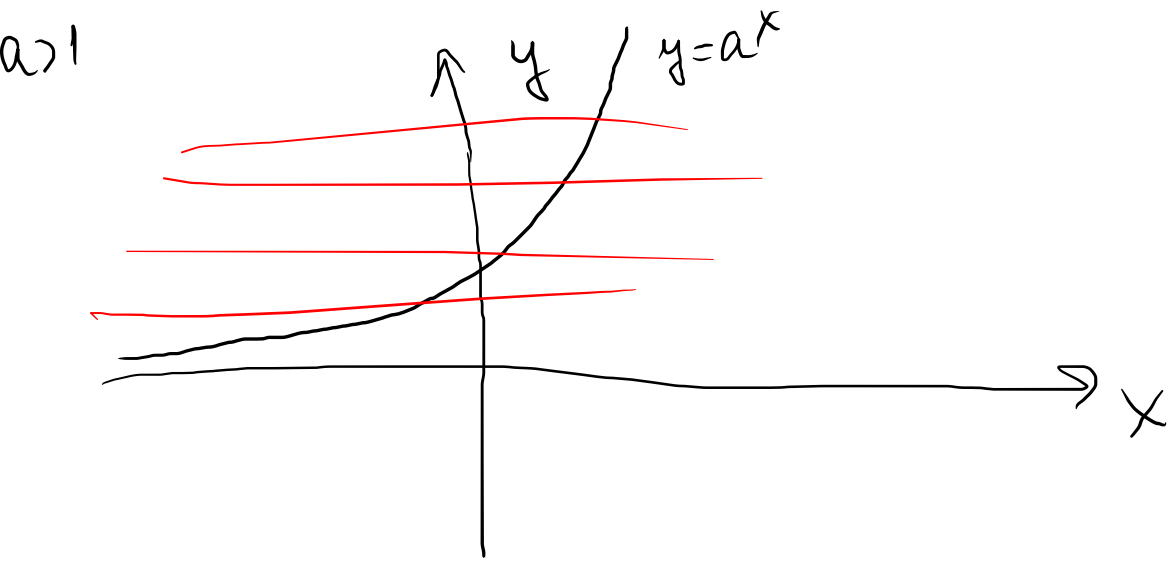


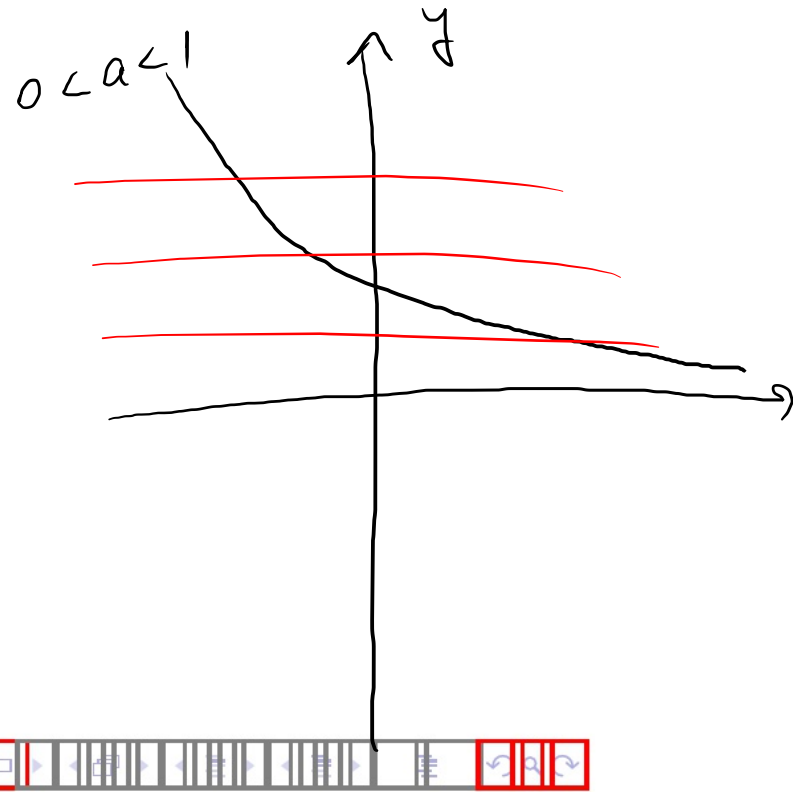
Eksponencijalna jednačina je oblika

$$a^x = b, \;\; a, b > 0, \;\; a \neq 1.$$

Kako je eksponencijalna funkcija injektivna, važi

$$a^x = a^y \quad \Leftrightarrow \quad x = y.$$





Primer: U skupu $\mathbb{R}$ rešiti jednačine:

1. $\left(\sqrt{3}\right)^{x^2-x} = 27$

$$\left(3^{\frac{1}{2}}\right)^{x^2-x} = 3^3$$

$$3^{\frac{1}{2}(x^2-x)} = 3^3$$

$$\frac{1}{2}(x^2-x) = 3 \quad /2$$

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$x_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \begin{matrix} 3 \\ -2 \end{matrix}$$

2. $8^x = 7^{x-1} + 7^x$

$8^x = 7^x \cdot 7^{-1} + 7^x$

$8^x = \frac{1}{7} 7^x + 7^x$

$8^x = \left(\frac{1}{7} + 1\right) \cdot 7^x$

$8^x = \frac{8}{7} 7^x \quad / : 7^x > 0$

$\frac{8^x}{7^x} = \frac{8}{7}$

$\left(\frac{8}{7}\right)^x = \frac{8}{7}$

$x = 1$

3. $3^{x+2} + 9^{x+1} = 810$

4. $4^{x^2+2} - 9 \cdot 2^{x^2+2} + 8 = 0$

$$\left(2^2\right)^{x^2+2} - 9 \cdot 2^{x^2+2} + 8 = 0$$

$$\left(2^{x^2+2}\right)^2 - 9 \cdot 2^{x^2+2} + 8 = 0$$

$2^{x^2+2} = t$

$$t^2 - 9t + 8 = 0$$

$$t_{1,2} = \frac{9 \pm \sqrt{81 - 32}}{2} = \frac{9 \pm 7}{2} \quad \begin{matrix} 1 \\ 8 \end{matrix}$$

$2^{x^2+2} = 1$

$2^{x^2+2} = 2^0$

$x^2 + 2 = 0$

$x \in \emptyset$

$2^{x^2+2} = 8$

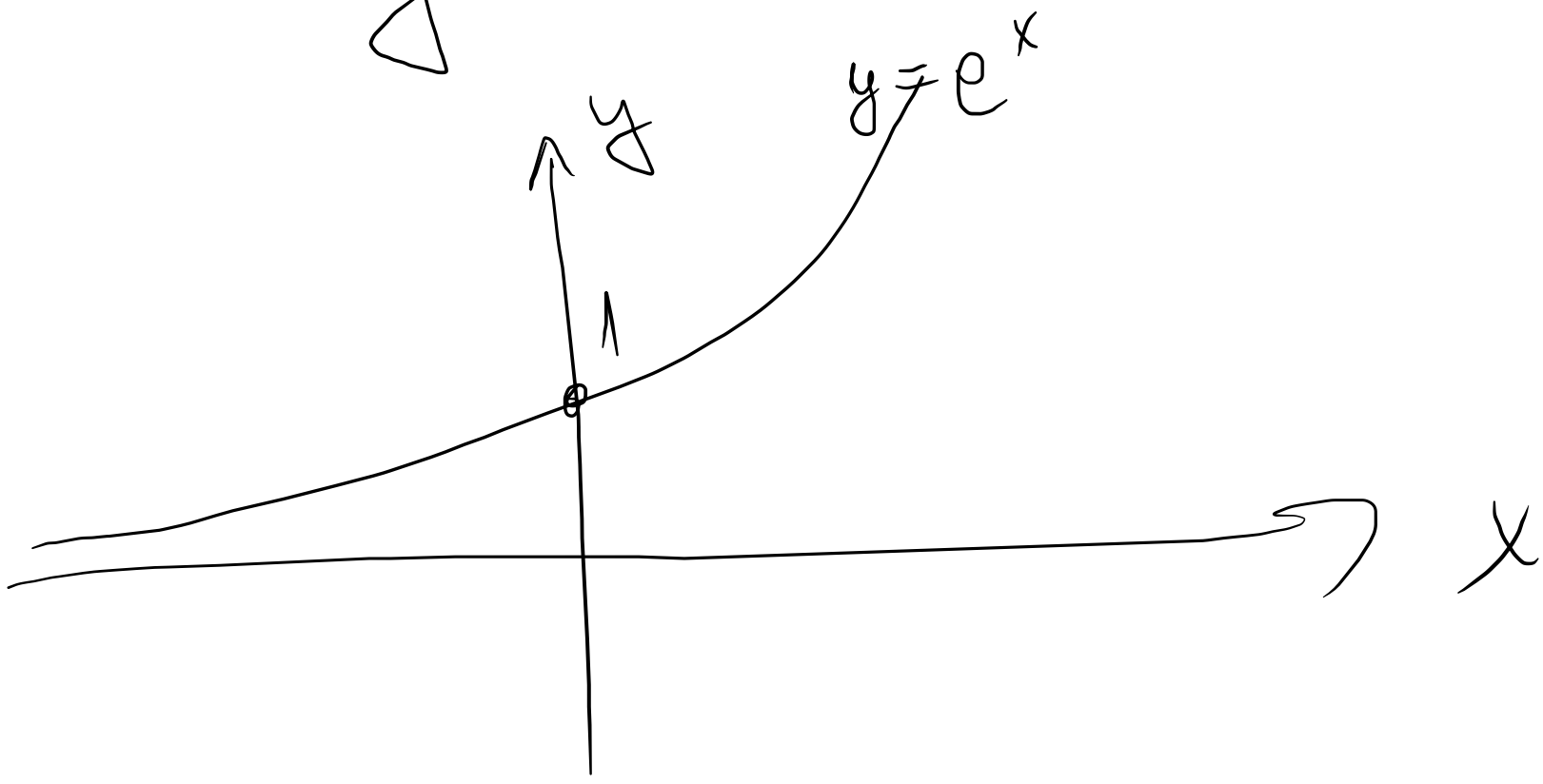
$2^{x^2+2} = 2^3$

$x^2 + 2 = 3$

$x^2 = 1$

$x = 1 \quad x = -1$

$y = e^x$



5. $9^x + 6^x = 2 \cdot 4^x$

Eksponencijalne nejednačine:

- ako je $a > 1$, onda važi:

$$a^{f(x)} \leq a^{g(x)} \Leftrightarrow f(x) \leq g(x)$$

- ako je $0 < a < 1$, onda važi:

$$a^{f(x)} \leq a^{g(x)} \Leftrightarrow f(x) \geq g(x).$$

Primer: Rešiti nejednačinu:

1. $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

$\frac{4}{5} < 1$ $\qquad x-1 > 4(1+\sqrt{x})$

2. $2^x + 2^{1-x} - 3 < 0$

$2^x + 2 \cdot 2^{-x} - 3 < 0$

$2^x + 2 \cdot \frac{1}{2^x} - 3 < 0 \quad / \cdot 2^x > 0$

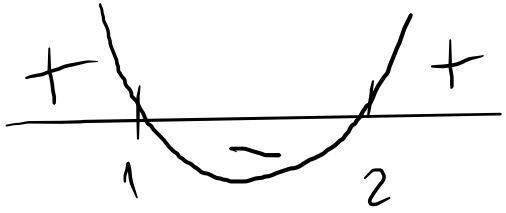
$2^{2x} + 2 - 3 \cdot 2^x < 0$

$2^x = t$

$t^2 - 3t + 2 < 0$

$t^2 - 3t + 2 = 0$

$t_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} < \begin{matrix} 2 \\ 1 \end{matrix}$



$t \in (1,2)$

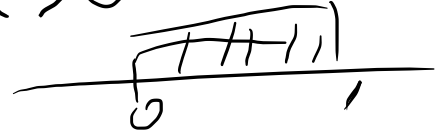
$2^x \in (1,2)$

$1 < 2^x < 2$

$2^x > 1 \wedge 2^x < 2$

$2^x > 2^0 \wedge 2^x < 2^1$

$x > 0 \wedge x < 1$



$x \in (0,1)$

3. $2^{4x+2} \cdot 4^{-x^2} - 3 \cdot 2^{2+2x-x^2} + 8 \leq 0$