Realne funkcije dve (više) realne promenljive - ekstremne vrednosti

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Funkcija $z = f(x, y)$ definisana u oblasti $D \subset \mathbb{R}^2$ ima lokalni ekstrem u tački (x, y) ako je $\triangle\, z$ istog znaka za sve dovoljno male vrednosti $\triangle\, x$ i $\triangle\, y$ ((x, y), $(x + \triangle\, x, y + \triangle\, y)$ pripadaju oblasti D), i to lokalni maksimum ako je $\triangle\, z < 0$, a lokalni minimum ako je $\triangle\, z > 0$.

Potreban uslov da diferencijabilna funkcija $z = f(x, y)$ ima lokalni ekstrem u tački $T(x_0, y_0) \in D$ jeste da je

$$\frac{\partial z}{\partial x}(x_0, y_0) = 0 \quad \text{i} \quad \frac{\partial z}{\partial y}(x_0, y_0) = 0.$$

Tačke u kojima je zadovoljen potreban uslov za lokalnu ekstremnu vrednost nazivaju se stacionarne tačke.

Dovoljani uslovi za postojanje lokalne ekstremne vrednosti funkcije $z = f(x, y)$ u tački T:

$\ast$ Neka je $T(x_0, y_0)$ stacionarna tačka funkcije $z = f(x, y)$. Neka u nekoj okolini tačke $T(x_0, y_0)$, uključujući i tu tačku, funkcija $z = f(x, y)$ ima neprekidne parcijalne izvode. Tada, ako je

1. $\left(d^2z\right)_T > 0$ za $(dx, dy) \neq (0, 0)$, funkcija $z = f(x, y)$ u tački $T(x_0, y_0)$ ima lokalni minimum;
2. $\left(d^2z\right)_T < 0$ za $(dx, dy) \neq (0, 0)$, funkcija $z = f(x, y)$ u tački $T(x_0, y_0)$ ima lokalni maksimum;
3. $\left(d^2z\right)_T$ menja znak za $(dx, dy) \neq (0, 0)$, funkcija $z = f(x, y)$ u tački $T(x_0, y_0)$ nema lokalne ekstreme.

$\ast$ Neka je $T(x_0, y_0)$ stacionarna tačka funkcije $z = f(x, y)$. Neka u nekoj okolini tačke $T(x_0, y_0)$, uključujući i tu tačku, funkcija $z = f(x, y)$ ima neprekidne parcijalne izvode i neka je

$$r = \frac{\partial^2 z}{\partial x^2}(x_0, y_0), \quad s = \frac{\partial^2 z}{\partial y^2}(x_0, y_0), \quad t = \frac{\partial^2 z}{\partial x \partial y}(x_0, y_0).$$

Tada, ako je

1. $rs - t^2 > 0$ i $r > 0$ (ili $s > 0$), funkcija $z = f(x, y)$ u tački $T(x_0, y_0)$ ima lokalni minimum;
2. $rs - t^2 > 0$ i $r < 0$ (ili $s < 0$), funkcija $z = f(x, y)$ u tački $T(x_0, y_0)$ ima lokalni maksimum;
3. $rs - t^2 < 0$, funkcija $z = f(x, y)$ u tački $T(x_0, y_0)$ nema lokalne ekstreme;
4. $rs - t^2 = 0$, potrebna su dalja ispitivanja.

Zadatak. Odrediti ekstremne vrednosti funkcije:

1. $z = x^2 + 2y^2 - 2x + 4y - 6;$

2. $z = xy;$

3. $z = \dfrac{x}{y} + \dfrac{2}{x} + 4y,\ x > 0,\ y > 0;$

D7

4. $z = -x^3 - y^3 + 3xy;$

ДЗ

5. $z = 4(x - y) - x^2 - y^2$;

DZ

6. $z = x^3 + y^3 - 9xy + 1;$

7. $z = x^3 + 3xy^2 - 12x;$

$$\frac{\partial z}{\partial x} = \boxed{3x^2 + 3y^2 - 12}$$

$$\frac{\partial z}{\partial y} = \boxed{6xy}$$

$$3x^2 + 3y^2 - 12 = 0$$
$$6xy = 0$$

$x = 0 \quad \vee \quad y = 0$

$3y^2 - 12 = 0 \qquad 3x^2 - 12 = 0$

$y^2 = 4 \qquad x^2 = 4$

$y = \pm 2 \qquad x = \pm 2$

$M_1(0,2) \qquad M_3(2,0)$

$M_2(0,-2) \qquad M_4(-2,0)$

STACIONARNE TACKE

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6x$$

II

$M_1(0,2)$

$r = \frac{\partial^2 z}{\partial x^2}(0,2) = 0$

$s = \frac{\partial^2 z}{\partial y^2}(0,2) = 0$

$t = \frac{\partial^2 z}{\partial x \partial y}(0,2) = 12$

$rs - t^2 = 0 - 144 < 0$

$\Rightarrow$ nema e.b.

$M_2(0,-2)$

$r = \frac{\partial^2 z}{\partial x^2}(0,-2) = 0$

$s = \frac{\partial^2 z}{\partial y^2}(0,-2) = 0$

$t = \frac{\partial^2 z}{\partial x \partial y}(0,-2) = -12$

$rs - t^2 = 0 - 144 < 0$

$\Rightarrow$ nema e.b.

$M_3(2,0)$

$r = \frac{\partial^2 z}{\partial x^2}(2,0) = 12$

$s = \frac{\partial^2 z}{\partial y^2}(2,0) = 12$

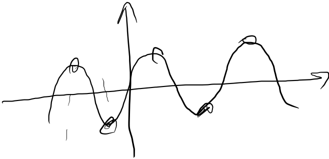
$t = \frac{\partial^2 z}{\partial x \partial y}(2,0) = 0$

$rs - t^2 = 144 - 0 > 0$

постоји е.в.

$r = 12 > 0 \Rightarrow$ локални минимум

$z_{min}(2,0) = 2^3 + 3 \cdot 2 \cdot 0^2 - 12 \cdot 2 = -16$



$\boxed{rs - t^2 > 0}$

POSTOJI E.V.

$r > 0$ lok min

$r < 0$ lok max

$M_4(-2,0)$

$r = \frac{\partial^2 z}{\partial x^2}(-2,0) = -12$

$s = \frac{\partial^2 z}{\partial y^2}(-2,0) = -12$

$t = \frac{\partial^2 z}{\partial x \partial y}(-2,0) = 0$

$rs - t^2 = 144 - 0 > 0$

постоји е.в.

$r = -12 < 0 \Rightarrow$ локални максимум

$z_{max}(-2,0) = (-2)^3 + 3 \cdot (-2) \cdot 0 - 12 \cdot (-2) = 16$

I

$$d^2z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2$$

$$= 6x\,dx^2 + 12y\,dx\,dy + 6x\,dy^2$$

$d^2z(0,2) = 24\underline{dx\,dy}$ - мења знак

$\Rightarrow$ нема е.б. у $M_1(0,2)$

$d^2z(0,-2) = -24\underline{dx\,dy}$ - мења знак

$\Rightarrow$ нема е.б. у $M_2(0,-2)$

$d^2z(2,0) = 12dx^2 + 12dy^2 = 12\underbrace{(dx^2+dy^2)}_{\geq 0} > 0$ $\Rightarrow$ локални минимум

$(dx,dy) \neq (0,0)$ $\qquad z_{min}(2,0) = -16$

$d^2z(-2,0) = -12dx^2 - 12dy^2 = -12\underbrace{(dx^2+dy^2)}_{\geq 0} < 0$ $\Rightarrow$ локални максимум

$(dx,dy) \neq (0,0)$ $\qquad z_{min}(-2,0) = 16$

8. $z = x^4 + y^4 - x^2 - 2xy - y^2;$

$$\frac{\partial z}{\partial x} = \boxed{4x^3 - 2x - 2y}$$

$$\frac{\partial z}{\partial y} = 4y^3 - 2x - 2y$$

$$4x^3 - 2x - 2y = 0 \quad /:2$$
$$4y^3 - 2x - 2y = 0 \quad /:2$$

$$2x^3 = x + y$$
$$2y^3 = x + y$$

$$\cancel{2}x^3 = \cancel{2}y^3$$
$$x^3 = y^3$$
$$\boxed{x = y}$$

$$2x^3 = x + x$$
$$\cancel{2}x^3 = \cancel{2}x \qquad \boxed{/:x} \ \text{NE!}$$

$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$\boxed{x = 0} \qquad x^2 = 1 \qquad \boxed{x = 1} \qquad \boxed{x = -1}$$

$M_1(0,0)$

$M_2(1,1)$

$M_3(-1,-1)$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 2$$
$$\frac{\partial^2 z}{\partial x \partial y} = -2$$
$$\frac{\partial^2 z}{\partial y^2} = 12y^2 - 2$$

$M_2(1,1)$

$$r = \frac{\partial^2 z}{\partial x^2}(1,1) = 10$$
$$s = \frac{\partial^2 z}{\partial y^2}(1,1) = 10$$
$$t = \frac{\partial^2 z}{\partial x \partial y}(1,1) = -2$$

I $M(0,0)$

$$r = \frac{\partial^2 z}{\partial x^2}(0,0) = -2$$
$$s = \frac{\partial^2 z}{\partial y^2}(0,0) = -2$$
$$t = \frac{\partial^2 z}{\partial x \partial y}(0,0) = -2$$

$$rs - t^2 = (-2)\cdot(-2) - (-2)^2 = 0$$

OVAJ NAČIN NE DAJE ODGOVOR !

$$rs - t^2 = 100 - 4 = 96 > 0$$

POSTOJI E.V. u M_2

$r = 10 > 0 \Rightarrow$ LOKALNI MINIMUM

$$z_{min}(1,1) = 1^4 + 1^4 - 1^2 - 2\cdot 1\cdot 1 - 1^2 = -2$$

$M_3(-1,-1)$

$r=\frac{\partial^2 z}{\partial x^2}(-1,-1)=10$

$s=\frac{\partial^2 z}{\partial y^2}(-1,-1)=10$

$t=\frac{\partial^2 z}{\partial x \partial y}(-1,-1)=-2$

$rs-t^2=100-4=96>0$

POSTOJI E.V. u M_3

$r=10>0 \Rightarrow$ LOKALNI MINIMUM

$z_{MIN}(-1,-1)=\cancel{1}+\cancel{1}-1-\cancel{2}-1=-2$

-2

I $\quad d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x \partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}dy^2=(12x^2-2)dx^2-4\,dx\,dy+(12y^2-2)dy^2$

NISMO MORALI

$d^2z(1,1)=10dx^2-4dx\,dy+10dy^2=2(5dx^2-2dx\,dy+5dy^2)=2(\underbrace{(dx-dy)^2+4dx^2+4dy^2}_{\geq 0})>0$

$(dx,dy)\neq(0,0)$

$d^2z(1,1)>0 \Rightarrow M_1(0,0)$ – LOKALNI MINIMUM

$d^2z(0,0)=-2dx^2-4dx\,dy-2dy^2=-2(dx^2+2dx\,dy+dy^2)=-2\underbrace{(dx+dy)^2}_{\geq 0}<0$ $\quad (dx,dy)\neq(0,0)$

$\Rightarrow$ LOKALNI MAKSIMUM $\quad z_{max}(0,0)=0$

PROVERITI DA LI U TAČKAMA $M_1(1,-1)$, $M_2(-\frac{1}{2},2)$, $M_3(\frac{1}{2},5)$, $M_4(2,-3)$ POSTOJI E.V. FUNKCIJE

9. $z = \ln(y - 2xy) + xy - x$; , 1 AKO POSTOJI ODREDITI JE

$$\frac{\partial z}{\partial x} = \frac{1}{y-2xy}\cdot(-2y) + y - 1$$

$$= \frac{1}{\cancel{y}(1-2x)}(-2\cancel{y}) + y - 1$$

$$= \boxed{\frac{-2}{1-2x} + y - 1}$$

$$\frac{\partial z}{\partial y} = \frac{1}{y-2xy}\cdot(1-2x) + x$$

$$= \frac{1}{y(1-\cancel{2x})}\cdot(1\cancel{-2x}) + x$$

$$= \frac{1}{y} + x$$

$$\frac{-2}{1-2x} + y - 1 = 0$$

$$\frac{1}{y} + x = 0$$

$M_1(1,-1)$: $\frac{1}{-1} + 1 = 0$ ✓

$-\frac{2}{1-2} - 1 - 1 = 0$ ✓

$M_2(-\frac{1}{2},2)$: $\frac{1}{2} - \frac{1}{2} = 0$ ✓

$-\frac{2}{1-\cancel{2}\cdot(-\frac{1}{\cancel{2}})} + 2 - 1 = -1 + 2 - 1 = 0$ ✓

~~$M_3: (\frac{1}{2},5)$~~: $\frac{1}{5} + \frac{1}{2} \neq 0$

~~$M_4(2,-3)$~~: $-\frac{1}{3} + 2 \neq 0$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-(-2)(-2)}{(1-2x)^2} = \frac{-4}{(1-2x)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 1$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{y^2}$$

$\boxed{M_1(1,-1)}$

$$r = \frac{\partial^2 z}{\partial x}(1,-1) = \frac{-4}{(1-2)^2} = -4$$

$$s = \frac{\partial^2 z}{\partial y^2}(1,-1) = -\frac{1}{(-1)^2} = -1$$

$$t = \frac{\partial^2 z}{\partial x \partial y}(1,-1) = 1$$

$$rs - t^2 = (-4)\cdot(-1) - 1 = 3 > 0$$

$\Rightarrow$ POSTOJI E.V u M_1

$r = -4 < 0 \Rightarrow$ LOKALNI MAX

$$z_{max}(1,-1) = \ln(-1 - 2\cdot 1(-1)) + 1\cdot(-1) - 1 = -2$$

$M_2\left(-\frac{1}{2}, 2\right)$

$$r = \frac{\partial^2 z}{\partial x^2}\left(-\frac{1}{2}, 2\right) = \frac{-4}{\left(1-2\left(-\frac{1}{2}\right)\right)^2} = -1$$

$$s = \frac{\partial^2 z}{\partial y^2}\left(-\frac{1}{2}, 2\right) = -\frac{1}{4}$$

$$t = \frac{\partial^2 z}{\partial x \partial y}\left(-\frac{1}{2}, 2\right) = 1$$

$$rs - t^2 = -1\cdot\left(-\frac{1}{4}\right) - 1 = \frac{1}{4} - 1 = -\frac{3}{4} < 0$$

$\Rightarrow$ NEMA E.V.

Dz

10. $z = 2x^3 + y^2 - 6x + 18y + 12;$

11. $z = e^{-x^2-y^2}$;

Dz

DZ

12. $z = -2x^2 - 2y^2 + 4x - 12y - 18;$

ДЗ

13. $z = x^2 + 2y^2 - 2x + 8y - 2xy + 10;$

14. $z = (x^2 + y)\cdot\sqrt{e^y};$

$$\frac{\partial z}{\partial x} = \boxed{2x\sqrt{e^y}}$$

$$\frac{\partial z}{\partial y} = \sqrt{e^y} + (x^2+y)\,\frac{1}{2\sqrt{e^y}}\cdot e^y$$

$$= \frac{2}{2}\sqrt{e^y} + (x^2+y)\,\frac{1}{2\cancel{\sqrt{e^y}}}\,\cancel{\sqrt{e^y}}\cdot\sqrt{e^y}$$

$$= \boxed{\frac{1}{2}\sqrt{e^y}\cdot(2 + x^2 + y)}$$

$$2x\sqrt{e^y} = 0$$

$$\frac{1}{2}\sqrt{e^y}\,(2 + x^2 + y) = 0$$

$$\boxed{X = 0} \qquad 2 + y = 0$$

$$\boxed{2 + x^2 + y = 0} \qquad \boxed{y = -2}$$

$$\boxed{M(0,-2)}$$

$$\frac{\partial^2 z}{\partial x^2} = 2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \cancel{2}x\,\frac{1}{\cancel{2}\sqrt{e^y}}\cdot e^y$$

$$= x\sqrt{e^y}$$

$\sqrt{e^y}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\left(\frac{1}{2\sqrt{e^y}}\cdot e^y\,(2 + x^2 + y) + \frac{2}{2}\sqrt{e^y}\right)$$

$$= \frac{1}{4}\sqrt{e^y}\,(2 + x^2 + y + 2) = \frac{1}{4}\sqrt{e^y}\,(4 + x^2 + y)$$

$$r = \frac{\partial^2 z}{\partial x^2}(0,-2) = 2\sqrt{e^{-2}}$$

$$s = \frac{\partial^2 z}{\partial y^2}(0,-2) = \frac{1}{4}\sqrt{e^{-2}}\,(4 + 0 - 2) = \frac{1}{2}\sqrt{e^{-2}}$$

$$t = \frac{\partial^2 z}{\partial x \partial y}(0,-2) = 0$$

$$rs - t^2 = \cancel{2}\sqrt{e^{-2}}\cdot\frac{1}{\cancel{2}}\sqrt{e^{-2}} - 0$$

$$= e^{-2} = \frac{1}{e^2} > 0$$

POSTOJI E.V. U M

$r = 2\sqrt{e^{-2}} > 0 \rightarrow$ LOKALNI MINIMUM

$$z_{min}(0,-2) = -2\sqrt{e^{-2}}$$