KOMPLEKSNI BROJEVI

i-imaginarna jedinica

$i^2 = -1$

$i^3 = i^2 i = -i$

$i^4 = i^3 i = 1$

$$i = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

$$i^5 = i$$
$$i^6 = -1$$
$$i^7 = -i$$
$$i^8 = 1$$

$$i^n = \begin{cases} 1, & \text{ako je n deljiv sa 4} \\ i, & \text{ako pri deljenju sa 4 dobijemo ostatak 1} \\ -1, & \text{ako pri deljenju sa 4 dobijemo ostatak 2} \\ -i, & \text{ako pri deljenju sa 4 dobijemo ostatak 3} \end{cases}$$

$\boxed{i^{4k} = 1}$

$i^{4k+1} = i$

$i^{4k+2} = -1$

$i^{4k+3} = -i$

Primeri:

$i^{18} = i^{16+2} = i^2 = -1$

$i^{79} = i^{76+3} = i^3 = -i$

$i^{61} = i^{60+1} = i$

$i^{108} = i^4 = 1$

$i^{18} = i^{4\cdot 4+2} = i^{4\cdot 4} \cdot i^2 = -1$

$i^{79} = i^{19\cdot 4+3} = i^{19\cdot 4} \cdot i^3 = i^3 = i^2 \cdot i = -i$

$i^{61} = i^{15\cdot 4+1} = i^{15\cdot 4} \cdot i = i$

$i^{108} = i^{27\cdot 4} = 1$

$79 : 4 = 19$
39
3

$61 : 4 = 15$
21
1

$108 : 4 = 27$
28

$i^{2024} = i^{4\cdot 506} = 1$

$2024 : 4 = 506$
20
24

57 : 4 = 14
17
1

1. Izračunati:

$i^{57} = i^{4 \cdot 14 + 1} = i^{4 \cdot 14} \cdot i = i$

$i^{2020} =$

$i^{1002} =$

$i^{99} =$

Algebarski oblik kompleksnog broja je

$$\boxed{z = a + bi}$$

$\boxed{a, b \in \mathbb{R}}$, i je imaginarna jedinica.

$\mathbb{C}$-skup kompleksnih brojeva

Realni deo kompleksnog broja $z = a + bi$ je $Re(z) = a$, a imaginarni deo je $Im(z) = b$

$Im(z) \neq bi$

$$z = \boxed{a} + \boxed{b}i \quad , \quad \underline{\underline{a, b \in \mathbb{R}}}$$

$$a = Re(z)$$

$$b = Im(z)$$

Primeri:

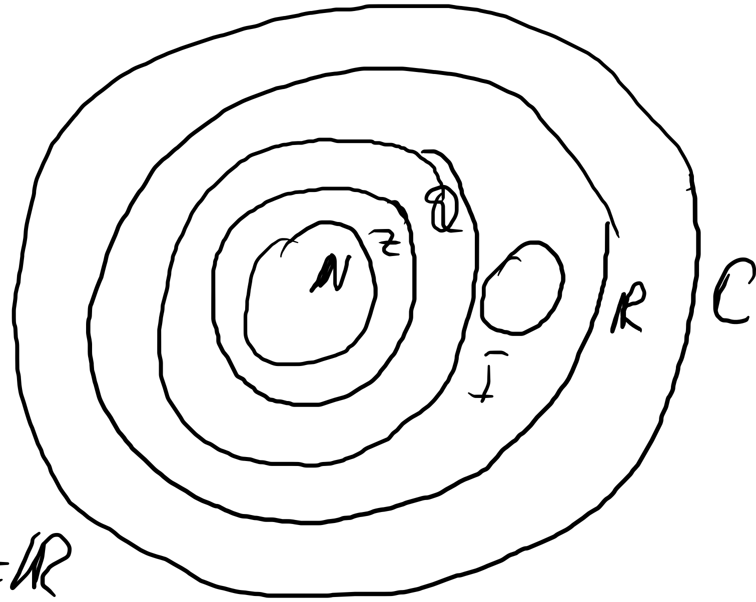
$z = 2 + 3i,\qquad Re(z) = 2, Im(z) = 3$

$z = 4 - 5i,\qquad Re(z) = 4, Im(z) = -5$

$z = \frac{5}{8} + \frac{3}{9}i,\qquad Re(z) = \frac{5}{8}, Im(z) = \frac{3}{9}$

$z = 2\qquad Re(z) = 2, Im(z) = 0$

$z = 3i,\qquad Re(z) = 0, Im(z) = 3$



$z = a + bi, \quad a, b \in \mathbb{R}$

$b = 0 \qquad \underline{\underline{z = a}}$

Dva kompleksna broja data u algebarskom obliku su jednaka ako i samo ako su im jednaki realni i imaginarni delovi, tj.

$$z_1 = z_2 \Leftrightarrow Re(z_1) = Re(z_2) \wedge Im(z_1) = Im(z_2)$$

$$a + bi = 5 + 6i$$

$$a = 5 \qquad b = 6$$

Konjugovano kompleksan broj broja $z = a + bi$ je $\bar{z} = a - bi$.

Primeri:

$z = 2 + 3i \Longrightarrow \bar{z} = 2 - 3i$

$z = 3 - 4i \Longrightarrow \bar{z} = 3 + 4i$

$z = 5 \Longrightarrow \bar{z} = 5$

$z = 7i \Longrightarrow \bar{z} = -7i$

$z = -3i \Longrightarrow \bar{z} = 3i$

$z = i - 1$

$\bar{z} = -i - 1$

$z = a + bi$

$\bar{z} = a - bi$

Operacije sa kompleksnim brojevima u algebarskom obliku

Neka je $z_1 = a + bi, z_2 = c + di$.

Tada je:

$$z_1 + z_2 = a + bi + c + di = a + c + (b + d)i$$

$$z_1 - z_2 = a + bi - (c + di) = a - c + (b - d)i$$

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = ac - bd + (ad + bc)i$$

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac - adi + bci - bdi^2}{c^2 - (di)^2} =$$

$$= \frac{ac + bd + (-ad + bc)i}{c^2 + d^2}$$

Primer:

1. Neka je $z_1 = 2 + 3i, z_2 = 1 + 2i$.

Tada je:

$z_1 + z_2 = 2 + 3i + 1 + 2i = 3 + 5i$

$z_1 - z_2 = 2 + 3i - (1 + 2i) = 2 + 3i - 1 - 2i = 1 + i$

$z_1 \cdot z_2 = (2 + 3i)(1 + 2i) = 2 + 4i + 3i + 6i^2 = 2 + 7i - 6 = -4 + 7i$

$$\frac{z_1}{z_2} = \frac{2+3i}{1+2i} = \frac{2+3i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{2-4i+3i-6i^2}{1-(2i)^2} = \frac{8-i}{5}$$

$z_1 = 5 + 2i$

$z_2 = -1 + i$

$z_1 + z_2 = 5 + 2i + (-1) + i = 4 + 3i$

$z_1 - z_2 = 5 + 2i - (-1 + i) = 5 + 2i + 1 - i = 6 + i$

$z_1 \cdot z_2 = (5+2i)(-1+i) = -5 + 5i - 2i + 2\overset{-1}{i^2}$

$= -5 + 5i - 2i - 2 = -7 + 3i$

$$\frac{z_1}{z_2} = \frac{5+2i}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{-5-5i-2i-2\overset{-1}{i^2}}{1+1} = \frac{-3-7i}{2} = -\frac{3}{2} - \frac{7}{2}i$$

$(A+B)(A-B) = A^2 - B^2$

2. Odrediti: $z_1 + z_2,\ z_1 - z_2,\ z_1 \cdot z_2, \frac{z_1}{z_2}$, ako je:

a) $z_1 = 1 + i, z_2 = 2 + 3i$

b) $z_1 = 1 - 2i, z_2 = i$

Moduo kompleksnog broja $z = a + bi$ je broj $|z| = \sqrt{a^2 + b^2}$.

Primer:

$z = 3 + 4i \qquad |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$z = 1 - 2i \qquad |z| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

$z = 3 \qquad |z| = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$

$z = -2i \qquad |z| = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2$

$$z = a + bi \Rightarrow |z| = \sqrt{a^2 + b^2}$$

3. Odrediti $|z|$ ako je

a) $z = -3 - 4i$

b) $z = \frac{1}{2} - \frac{3}{4} i$

c) $z = 4i$

a) $|z| = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = 5$

b) $|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{3}{4}\right)^2} = \sqrt{\frac{1}{4} + \frac{9}{16}} = \sqrt{\frac{13}{16}} = \frac{\sqrt{13}}{4}$

c) $|z| = \sqrt{0^2 + 4^2} = 4$

4. Dati su kompleksni brojevi: $z_1 = 2 + 3i, z_2 = -4 + 5i$. Odrediti:

a) $|z_1|, |z_2|, |\overline{z_1}|, |\overline{z_2}|$,

b) $|z_1 + z_2|$

c) $|z_1 - z_2|$

d) $|z_1 z_2|$

e) $\left|\frac{z_1}{z_2}\right|$

f) $|3z_1 - 2z_2|$

$$z_1 = 2+3i$$
$$\overline{z_1} = 2-3i$$

a) $|z_1| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$

$|z_2| = \sqrt{(-4)^2+5^2} = \sqrt{16+25} = \sqrt{41}$

$|\overline{z_1}| = \sqrt{2^2+(-3)^2} = \sqrt{13}$

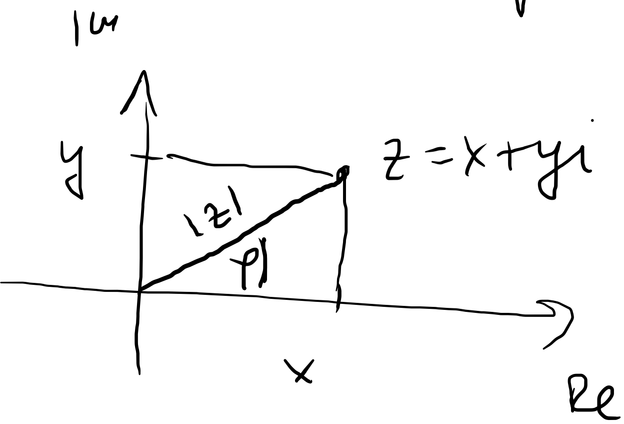
$|\overline{z_2}| = \sqrt{41}$

$$\underline{|z| = |\overline{z}|}$$

$z_1 + z_2 = 2+3i + (-4+5i) = -2+8i$

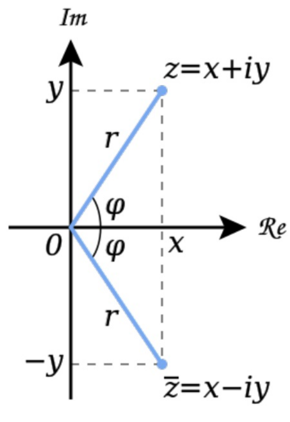
$|z_1+z_2| = \sqrt{(-2)^2+8^2} = \sqrt{4+64} = \sqrt{68}$

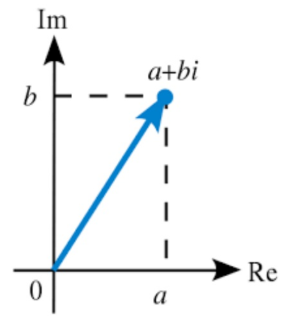
$z = x + yi$



Kompleksna (Gausova) ravan

Svaki kompleksan broj se može jednoznačno predstaviti kao tačka u kompleksnoj ravni. Kompleksna ravan je određena realnom (Re) i imaginarnom (Im) osom koje dele ravan na četiri kvadranta.





Moduo kompleksnog broja z je rastojanje kompleksnog broja z od koordinatnog početka.

Argument kompleksnog broja z, u oznaci $\arg(z) = \varphi$ je ugao koji poluprava $0z$ zaklapa sa pozitivnim smerom realne ose. Svakom kompleksnom broju $z \neq 0$ odgovara samo jedna vrednost φ iz intervala $(-\pi, \pi]$.

~~5~~. Odrediti argument za: $3, 3i, -3, 1+i, \sqrt{3}-i$

6. Naći realni i imaginarni deo kompleksnog broja $z = \frac{(1-i)^{12}}{(1+i)^5}$.

$$z = \frac{(1-i)^{12}}{(1+i)^5} = \frac{\left((1-i)^2\right)^6}{\left((1+i)^2\right)^2(1+i)} = \frac{(1-2i+i^2)^6}{(1+2i+i^2)^2(1+i)}$$

$$= \frac{(-2i)^6}{(2i)^2(1+i)} = \frac{\cancel{64}^{16}\, i^{\cancel{6}\,4}}{\cancel{4}\cdot \cancel{i^2}\cdot(1+i)} = \frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{1-i}{1+1} = \frac{1}{2}-\frac{1}{2}i$$

7. Neka je $z_1 = -4i, z_2 = -3 + 5i$. Odrediti: $Re(z_1),\ Re(z_2),\ Im(z_1),$ $Im(z_2),\ \ |z_1|, |z_2|, \overline{z_1}, \overline{z_2}, z_1 + z_2, z_1 - z_2, z_1 z_2, \frac{z_1}{z_2}, \cancel{arg(z_1)}$

8. Neka je $z = (2-3i)(1-5i)$. Odrediti: $Re(z)$, $Im(z)$, $\overline{z}$, $|z|$, $\frac{z}{i}$.

$$z = (2-3i)(1-5i) = 2-10i-3i+15i^2$$
$$= 2-13i-15$$
$$= -13-13i$$

$$Re(z) = -13$$

$$Im(z) = -13$$

$$\overline{z} = -13+13i$$

$$|z| = \sqrt{(-13)^2+(-13)^2} = \sqrt{169+169}$$

$$\frac{z}{i} = \frac{-13-13i}{i} \cdot \frac{-i}{-i} = \frac{13i+13i^2}{1} = -13+13i$$

9.Dati su kompleksni brojevi:

$$z_1 = 5 \cdot \frac{3+i}{2-i} + i^{17} - (1+2i)^2 \text{ i } z_2 = (1+i)^3 - 4 \cdot \frac{3}{1+i} - i^{103}.$$

Izračunati: $|\overline{z_1}|$, $|z_1 + \overline{z_2}|$, $Re(z_1 - z_2)$.

$$z_1 = 5 \cdot \frac{3+i}{2-i} + i^{17} - (1+2i)^2$$

$$= 5 \cdot \frac{3+i}{2-i} \cdot \frac{2+i}{2+i} + i^{4\cdot 4+1} - (1+4i+4i^2)$$

$$= 5 \, \frac{6+3i+2i+\overset{-1}{i^2}}{4+1} + \overset{1}{i^{4\cdot 4}} \, i - (1+4i-4)$$

$$= \cancel{5} \, \frac{5+5i}{\cancel{5}} + i + 3 - 4i$$

$$= 8 + 2i$$

10. Dat je kompleksan broj:

$z = (1+i)^2 + 2\overline{(3-i)} + \frac{4+2i}{1-i} + i(3-6i)$.

Izračunati: $Re(z+2)$ i $Im(\bar{z}-2)$.

$$\overline{(3-i)} = 3+i$$

11. Izračunati vrednost izraza

a) $\frac{z-\bar{z}}{1+z\bar{z}}$ ako je $z = 2 + i$ b) $\frac{z+\bar{z}}{2z+3}$ ako je $z = \frac{i+1}{2}$