

Aritmetička i geometrijska progresija

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ARITMETIČKI NIZ (PROGRESIJA)

Niz $a_1, a_2, \dots, a_n, \dots$ je aritmetički niz sa diferencijom (razlikom) d ako je

$$a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = d$$

$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & \dots \\ 5, & 8, & 11, & 14, & 17, & \dots \end{matrix}$
 $d = 3$

Za n -ti član niza važi

$$a_n = a_1 + (n-1)d$$

kao i da je

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Zbir prvih n članova aritmetičkog niza je

$$S_n = n \frac{a_1 + a_n}{2} = \frac{n}{2} (2a_1 + (n-1)d)$$

$$14 = 5 + (4-1) \cdot 3$$

$$a_6 = 5 + (6-1) \cdot 3$$

$$= 5 + 5 \cdot 3 = 20$$

$$a_{33} = 5 + (33-1) \cdot 3$$

$$= 5 + 32 \cdot 3 = 101$$

1. Tri broja čine aritmetičku progresiju. Njihov zbir je 6, a zbir njihovih kvadrata je 110. Koji su to brojevi?

$$a_1, a_2, a_3$$

$$a_1 + a_2 + a_3 = 6$$

$$a_1^2 + a_2^2 + a_3^2 = 110$$

$$a_1$$

$$a_2 = a_1 + d$$

$$a_3 = a_1 + 2d$$

$$a_1 + a_1 + d + a_1 + 2d = 6$$

$$3a_1 + 3d = 6$$

$$a_1 + d = 2$$

$$d = 2 - a_1$$

$$a_1^2 + (a_1 + d)^2 + (a_1 + 2d)^2 = 110$$

$$a_1^2 + (a_1 + 2 - a_1)^2 + (a_1 + 2(2 - a_1))^2 = 110$$

$$a_1^2 + 4 + (4 - a_1)^2 = 110$$

$$a_1^2 + 4 + 16 - 8a_1 + a_1^2 = 110$$

$$2a_1^2 - 8a_1 - 90 = 0 \quad | :2$$

$$a_1^2 - 4a_1 - 45 = 0$$

$$a_1^2 - 4a_1 - 45 = 0$$

$$\begin{array}{r} 14 \cdot 14 \\ \underline{56} \\ 14 \\ \underline{196} \end{array}$$

$$a_{1,2} = \frac{4 \pm \sqrt{16 + 180}}{2} = \frac{4 \pm 14}{2} = \begin{cases} 9 \\ -5 \end{cases}$$

$$a_1 = 9 \quad \checkmark$$

$$d = 2 - 9 = -7$$

$$a_2 = 9 - 7 = 2$$

$$a_3 = 2 - 7 = -5$$

$$9, 2, -5$$

$$a_1 = -5$$

$$d = 2 + 5 = 7$$

$$a_2 = -5 + 7 = 2$$

$$a_3 = 2 + 7 = 9$$

$$-5, 2, 9$$

$$d = 2 - a_1$$

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2. Cifre tricifrenog broja čine aritmetičku progresiju. Poslednja cifra jednaka je zbiru prve dve, a za jedan je veća od njihovog proizvoda. Koji je to broj?

 $a_1 a_2 a_3$

$$a_3 = a_1 + a_2$$

$$a_3 = a_1 a_2 + 1$$

$$a_1 + 2d = a_1 + a_1 + d$$

$$a_1 + 2d = a_1 (a_1 + d) + 1$$

$$a_1 = d$$

$$a_1 + 2d = a_1^2 + a_1 d + 1$$

$$a_1 + 2a_1 = a_1^2 + a_1^2 + 1$$

$$3a_1 = 2a_1^2 + 1$$

$$2a_1^2 - 3a_1 + 1 = 0$$

$$a_1$$

$$a_2 = a_1 + d$$

$$a_3 = a_1 + 2d$$

$$a_{1/2} = \frac{3 \pm \sqrt{9 - 4}}{4} = \frac{3 \pm 1}{4} \left(\frac{1}{2} \right)$$

$$a_1 = 1$$

$$d = 1$$

$$a_2 = 2$$

$$a_3 = 3$$

~~$a_1 = \frac{1}{2}$~~ jer je a_1 cifra broja

Rezultat je

$$\boxed{123}$$

3. Koliko članova niza 1, 4, 7, ... treba sabrati da se dobije zbir 210?

1, 4, 7, ...

$$S_n = 210$$

n ?

$$\left. \begin{aligned} S_n &= n \frac{a_1 + a_n}{2} \\ a_n &= a_1 + (n-1)d \end{aligned} \right\} S_n = n \frac{a_1 + a_1 + (n-1)d}{2}$$

$$S_n = \frac{n}{2} (2a_1 + (n-1)d)$$

$$\left. \begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 7 \end{aligned} \right\} d = 4 - 1 = 7 - 4 = 3$$

$$210 = \frac{n}{2} (2 \cdot 1 + (n-1) \cdot 3) / 2$$

$$420 = n(2 + 3n - 3)$$

$$420 = 3n^2 - n$$

$$3n^2 - n - 420 = 0$$

$$\begin{array}{r} 420 : 12 \\ \underline{420} \\ 0 \end{array}$$

$$\begin{array}{r} 5041 \\ \underline{4141} \\ 1200 \\ \underline{1281} \\ 7171 \\ \underline{7171} \\ 000 \\ \underline{400} \\ 5041 \end{array}$$

$$\begin{aligned} n_{1,2} &= \frac{1 \pm \sqrt{1 + 5040}}{6} \\ &= \frac{1 \pm 71}{6} = \left\langle \begin{array}{l} 12 \\ \cancel{\frac{70}{3}} \end{array} \right. \end{aligned}$$

$n \in \mathbb{N}$

$$n = 12$$

GEOMETRIJSKI NIZ (PROGRESIJA)

Niz $b_1, b_2, \dots, b_n, \dots$ je geometrijskiniz sa količnikom q ako je

$$\frac{b_2}{b_1} = \frac{b_3}{b_2} = \dots = \frac{b_n}{b_{n-1}} = \dots = q$$

Za n -ti član niza važi

$$b_n = b_1 q^{n-1}$$

kao i da je

$$b_n^2 = b_{n-1} \cdot b_{n+1}$$

Zbir prvih n članova geometrijskog niza je

$$S_n = b_1 \frac{1 - q^n}{1 - q} = b_1 \frac{q^n - 1}{q - 1}$$

Zbir prvih n članova geometrijskog niza je konačan samo ako je $|q| < 1$

i iznosi
$$S_n = \frac{b_1}{1 - q}$$

$$b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5 \\ 2, 6, 18, 54, 162, \dots$$

$$q = 3$$

$$6 = 2 \cdot 3^{2-1}$$

$$162 = 2 \cdot 3^{5-1}$$

$$b_{33} = 2 \cdot 3^{33-1}$$

1. Odrediti geometrijski niz ako je $b_3 + b_4 = 180$, a $b_5 + b_6 = 1620$.

$$b_3 + b_4 = 180$$

$$b_5 + b_6 = 1620$$

$$b_1 \cdot q^2 + b_1 \cdot q^3 = 180$$

$$b_1 \cdot q^4 + b_1 \cdot q^5 = 1620$$

$$b_1 q^2 (1+q) = 180$$

$$b_1 q^4 (1+q) = 1620$$

$$\frac{b_1 q^4 (1+q)}{b_1 q^2 (1+q)} = \frac{1620}{180}$$

$$q^2 = 9$$

$$b_1$$

$$b_2 = b_1 q$$

$$b_3 = b_1 \cdot q^2$$

$$b_4 = b_1 \cdot q^3$$

$$b_5 = b_1 \cdot q^4$$

$$b_6 = b_1 \cdot q^5$$

$$q = \pm 3$$

$$I \quad q = 3$$

$$b_1 \cdot 3^2 (1+3) = 180$$

$$b_1 \cdot 9 \cdot 4 = 180$$

$$b_1 = \frac{180}{36}$$

$$b_1 = 5$$

$$II \quad q = -3$$

$$b_1 \cdot (-3)^2 (1-3) = 180$$

$$b_1 \cdot 9 \cdot (-2) = 180$$

$$b_1 = \frac{180}{-18}$$

$$b_1 = -10$$

$$-10, 30, -90, \dots$$

$$5, 15, 45, \dots$$

2. Broj 26 podeliti na tri dela, koji predstavljaju uzastopne članove u geometrijskom nizu, tako da proizvod tih delova bude 216.

$$\begin{aligned} b_1 + b_2 + b_3 &= 26 \\ b_1 \cdot b_2 \cdot b_3 &= 216 \end{aligned}$$

$$b_1 + b_1 q + b_1 q^2 = 26$$

$$b_1 \cdot b_1 \cdot q \cdot b_1 \cdot q^2 = 216$$

$$b_1 (1 + q + q^2) = 26$$

$$b_1^3 q^3 = 216$$

$$b_1 (1 + q + q^2) = 26$$

$$(b_1 q)^3 = 6^3$$

$$\begin{aligned} b_1 & \\ b_2 &= b_1 q \\ b_3 &= b_1 q^2 \end{aligned}$$

$$b_1 (1 + q + q^2) = 26$$

$$b_1 \cdot q = 6$$

$$\frac{\cancel{b_1} (1 + q + q^2)}{\cancel{b_1} q} = \frac{26}{6} = \frac{13}{3}$$

$$3 + 3q + 3q^2 = 13q$$

$$3q^2 - 10q + 3 = 0$$

$$q_{1,2} = \frac{10 \pm \sqrt{100 - 36}}{6} = \frac{10 \pm 8}{6} = \left\langle \begin{array}{l} 3 \\ \frac{1}{3} \end{array} \right.$$

$$\text{I} \quad g = 3$$

$$b_1 \cdot g = 6 \quad b_1 = \frac{6}{3}$$

$$b_1 = 2$$

$$2, 6, 18$$



$$2 + 6 + 18 = 26$$

$$\text{II} \quad g = \frac{1}{3}$$

$$b_1 = \frac{6}{\frac{1}{3}} = 18$$

$$b_2 = 6$$

$$b_3 = 2$$

3. Zbir prvih n članova geometrijskog niza, čiji je količnik 2, iznosi 15, a zbir sledećih n članova je 240. Izračunati n i b_{2n} .

$$q = 2$$

$$S_n = 15$$

$$S_{n'} = 240$$

$$\underbrace{b_1, b_2, b_3, \dots, b_n}_{S_n = 15}, \underbrace{b_{n+1}, b_{n+2}, \dots, b_{2n}}_{S_{n'} = 240}$$

$$S_n = b_1 \frac{1 - 2^n}{1 - 2}$$

$$15 = b_1 \frac{1 - 2^n}{1 - 2}$$

$$\underline{\underline{15 = b_1 (2^n - 1)}}$$

$$S_{n'} = b_{n+1} \frac{1 - 2^n}{1 - 2}$$

$$240 = b_{n+1} \frac{1 - 2^n}{1 - 2}$$

$$\underline{\underline{240 = b_{n+1} (2^n - 1)}}$$

$$b_{n+1} = b_1 \cdot 2^n$$

$$\underline{\underline{240 = b_1 2^n (2^n - 1)}}$$

$$\frac{\cancel{b_1} \cdot 2^n \cdot (\cancel{2^n - 1})}{\cancel{b_1} (\cancel{2^n - 1})} = \frac{\cancel{240} \overset{48}{16}}{\cancel{15} \times 1}$$

$$2^n = 16$$

$$2^n = 2^4$$

$$\boxed{n = 4}$$

$$b_1 (2^n - 1) = 15$$

$$b_1 (2^4 - 1) = 15$$

$$b_1 \cdot 15 = 15$$

$$\boxed{b_1 = 1}$$

$$\begin{aligned} b_{2n} &= b_1 \cdot 2^{2n-1} \\ &= 1 \cdot 2^{2 \cdot 4 - 1} \\ &= 2^7 \end{aligned}$$

4. Drugi, peti i četrnaesti član aritmetičkog niza, čija je diferencija 10, ujedno su prva tri člana geometrijskog niza. Odrediti taj aritmetički i geometrijski niz.

$$\left. \begin{array}{l} a_2 = b_1 \\ a_5 = b_2 \\ a_{14} = b_3 \end{array} \right\} \rightarrow \left. \begin{array}{l} a_2 = a_1 + d \\ a_5 = a_1 + 4d \\ a_{14} = a_1 + 13d \end{array} \right\}$$

$$b_2^2 = b_1 \cdot b_3$$

$$a_5^2 = a_2 \cdot a_{14}$$

$$\boxed{d=10}$$

$$(a_1 + 4d)^2 = (a_1 + d)(a_1 + 13d)$$

$$\cancel{a_1^2} + \cancel{8da} + \underline{16d^2} = \cancel{a_1^2} + \underline{13a_1d} + \underline{a_1d} + \underline{13d^2}$$

$$3d^2 = 6a_1d \quad | :d = 10 \neq 0$$

$$300 = 60a_1$$

$$3d = 6a_1$$

$$30 = 6a_1$$

$$\boxed{a_1 = 5}$$

ARITMETIČKI
5, 15, 25, ...

$$a_2 = 5 + d$$

$$a_2 = 15$$

$$a_3 = 5 + 2 \cdot 10$$

$$= 25$$

GEOMETRIJSKI

$$b_1 = a_2 = 15$$

$$b_2 = a_5 = a_1 + 4d$$

$$= 5 + 4 \cdot 10 = 45$$

$$b_3 = a_{14} = a_1 + 13d$$

$$= 5 + 13 \cdot 10 = 135$$

5. Tri broja čiji je zbir 26 čine geometrijski niz. Uveća li se srednji član za 4, dobija se aritmetički niz. Koji su to brojevi?

$$b_1 + b_2 + b_3 = 26$$

$$\begin{cases} a_1 = b_1 \\ a_2 = b_2 + 4 \\ a_3 = b_3 \end{cases}$$

$$a_2 = \frac{a_1 + a_3}{2}$$

$$b_2 + 4 = \frac{b_1 + b_3}{2} \quad | \cdot 2$$

$$2 \cdot b_1 q + 8 = b_1 + b_1 q^2$$

$$b_1 - 2b_1 q + b_1 q^2 = 8$$

$$\boxed{b_1(1 - 2q + q^2) = 8}$$

$$\begin{cases} b_1 \\ b_2 = b_1 q \\ b_3 = b_1 q^2 \end{cases}$$

$$b_1 + b_2 + b_3 = 26$$

$$b_1 + b_1 q + b_1 q^2 = 26$$

$$\boxed{b_1(1 + q + q^2) = 26}$$

$$\frac{b_1(1 + q + q^2)}{b_1(1 - 2q + q^2)} = \frac{26}{8} \quad | \cdot 4$$

$$4 + 4q + 4q^2 = 13 - 26q + 13q^2$$

$$9q^2 - 30q + 9 = 0 \quad | : 3$$

$$3q^2 - 10q + 3 = 0 \quad q_{1,2} = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$q_1 = 3$$

$$q_2 = \frac{1}{3}$$

$$= \frac{10 \pm 8}{6} = \frac{18}{6} = 3$$

$$b_1(1+q+q^2)=26$$

$$\text{I} \quad q=3$$

$$b_1(1+3+9)=26$$

$$b_1 \cdot 13 = 26$$

$$\boxed{b_1=2}$$

$$b_2 = b_1 \cdot q = 2 \cdot 3 = 6$$

$$b_3 = b_1 \cdot q^2 = 2 \cdot 9 = 18$$

geometrische: 2, 6, 18

arithmetische: 2, 10, 18

14

18, 6, 2

18, 10, 2

$$\text{II} \quad q = \frac{1}{3}$$

$$b_1(1 + \frac{1}{3} + \frac{1}{9}) = 26$$

$$b_1 \frac{9+3+1}{9} = 26$$

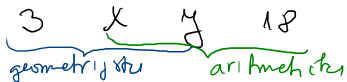
$$b_1 \frac{13}{9} = 26$$

$$\boxed{b_1=18}$$

$$b_2 = b_1 \cdot q = 18 \cdot \frac{1}{3} = 6$$

$$b_3 = b_1 \cdot q^2 = 18 \cdot \frac{1}{9} = 2$$

6. Između brojeva 3 i 18 su umetnuta dva cela broja . Prva tri broja su prva tri člana geometrijskog niza, a poslednja tri prva tri člana aritmetičkog niza. Koji su to brojevi?



$$x^2 = 3y$$

$$y = \frac{18+x}{2}$$

$$y = 9 + \frac{x}{2}$$

$$x^2 = 27 + \frac{3}{2}x \quad | \cdot 2$$

$$2x^2 - 3x - 54 = 0$$

$$x_{1,2} = \frac{3 \pm \sqrt{9+432}}{4} = \frac{3 \pm \sqrt{441}}{4} = \frac{3 \pm 21}{4} \quad \begin{matrix} 6 \\ -\frac{18}{4} \end{matrix}$$

$$x = 6$$

$$y = 9 + \frac{6}{2} = 9 + 3 = 12$$

$$3 \quad \boxed{6 \quad 12} \quad 18$$