Logaritamska funkcija, jednačine i nejednačine

17. април 2024.

Logaritamska funkcija

$\log_5 x$ ✓

$\log_{-5} x$ ✗

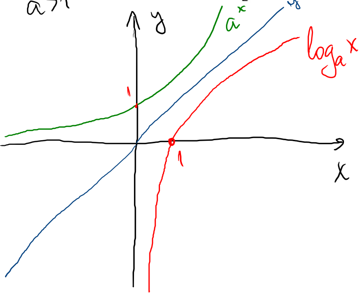
$$y = \log_a x, \quad a > 0, \quad a \neq 1,$$

$\boxed{\log_e x = \ln x}$

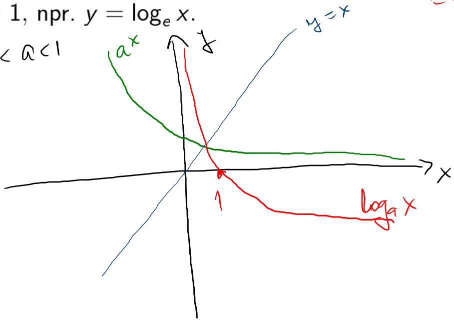
$\log x = \log_{10} x = \lg x$

definisana je za sve $x > 0$. Presečna tačka sa x-osom je $(1, 0)$. Logaritamska funkcija je opadajuća za osnovu $0 < a < 1$, npr. $y = \log_{\frac{1}{2}} x$, a rastuća za $a > 1$, npr. $y = \log_e x$.

$a > 1$



$0 < a < 1$



Logaritamska jednačina je oblika

$$\log_a x = b, \quad a, b > 0, \quad a \neq 1$$

Logaritamska i eksponencijalna funkcija sa istom osnovom su međusobno inverzne funkcije, pa imamo

$$\boxed{\log_a x = b \quad \Leftrightarrow \quad x = a^b.}$$

$\log_2 x = 1$

$x = 2^1$

$\log_2 4 = ? = 2$

$4 = 2^{\boxed{2}?}$

Kako je logaritamska funkcija injektivna, važi

$$\boxed{\log_a x = \log_a y \quad \Leftrightarrow \quad x = y.}$$

$\log_3 x = \log_3 (2x+5)$

$x = 2x+5$

$-x = 5$

$x = -5$

$x > 0$

$2x+5 > 0$

Logaritamske nejednačine:

- ako je $a > 1$, onda važi:

$$\log_a f(x) \leq \log_a g(x) \Leftrightarrow f(x) \leq g(x)$$

- ako je $0 < a < 1$, onda važi:

$$\log_a f(x) \leq \log_a g(x) \Leftrightarrow f(x) \geq g(x).$$

Osobine logaritama:

$x, y > 0,\ \ a, b, c \in \mathbb{R}^+ \setminus \{1\}$

1) $\log_a(x \cdot y) = \log_a x + \log_a y$
2) $\log_a \frac{x}{y} = \log_a x - \log_a y$
3) $\log_a x^n = n \cdot \log_a x$
4) $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$
5) $\log_a 1 = 0 \ \wedge \ \log_a a = 1$
6) $\log_a b = \frac{\log_c b}{\log_c a}$
7) $\log_a b = \frac{1}{\log_b a}$
8) $\log_{a^r} x = \frac{1}{r} \log_a x$
9) $a^{y \log_a x} = \left(a^{\log_a x}\right)^y = x^y$

$\log_2 3 = \frac{\log 3}{\log 2}$

$\log_2 3 = \frac{1}{\log_3 2}$

$\log_2 3 + \log_2 5 = \log_2(3 \cdot 5)$

$\log_2 3 - \log_2 5 = \log_2 \frac{3}{5}$

$\log_2 3^5 = 5 \cdot \log_2 3$

$\log_a 1 = 0$

$\log_a a = 1$

$\log_{2^5} 3 = \frac{1}{5} \log_2 3$

$a^{\log_a x} = x$

$3^{\log_3 x} = x$

Zadaci:

1. Rešiti jednacine:

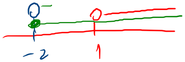
   1.1 $\log(x-1) + 2\log\sqrt{x+2} = 1$

USLOVI:

$x-1>0 \quad \wedge \quad \sqrt{x+2} > 0 \quad \wedge \quad x+2 \geq 0$

$x>1 \quad \wedge \quad x+2 \neq 0 \qquad x \geq -2$

$x \neq -2$



$\boxed{x \in (1, +\infty)}$

$\log(x-1) + 2\log(x+2)^{\frac{1}{2}} = 1$

$\log(x-1) + \cancel{2} \cdot \frac{1}{\cancel{2}} \log(x+2) = 1$

$\log(x-1) + \log(x+2) = 1$

$\log_{10}((x-1)(x+2)) = 1$

$(x-1)(x+2) = 10$

$x^2 + x - 2 = 10$

$x^2 + x - 12 = 0$

$x_{1,2} = \frac{-1 \pm \sqrt{1+48}}{2} = \frac{-1 \pm 7}{2} = \begin{cases} 3 \\ -4 \end{cases}$

$\boxed{x_1 = 3} \in (1, +\infty)$ ✓

$x = -4 \notin (1, +\infty)$

1.2 $\log_2^2 x + 2\log_2\sqrt{x} = 2$

USLOVI: $x>0 \wedge \sqrt{x}>0 \wedge x\geq 0$

$x\neq 0$

$\boxed{x\in(0,\infty)}$

$$(\log_2 x)^2 + 2\log_2 x^{\frac{1}{2}} = 2$$

$$(\log_2 x)^2 + \cancel{2}\cdot\frac{1}{\cancel{2}}\log_2 x = 2$$

$$(\log_2 x)^2 + \log_2 x = 2$$

$\log_2 x = t \qquad t^2+t-2=0$

$$t_{1,2} = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} < \begin{matrix}1\\-2\end{matrix}$$

$\boxed{\log_a x = b \Leftrightarrow x = a^b}$

$t_1 = 1$

$\log_2 x = 1$

$x = 2^1$

$\boxed{x=2}\in(0,\infty)$ ✓

$\log_2 x = \log_2 2$

$\boxed{x=2}$

$t=-2$

$\log_2 x = -2$

$x = 2^{-2}$

$\boxed{x=\frac{1}{4}}\in(0,\infty)$ ✓

$\log_2 x = -2\cdot\log_2 2$

$\log_2 x = \log_2 2^{-2}$

$x = 2^{-2}$

1.3 $\log_2(7+2^{-x}) = x+3$

Uslov: $7+\underbrace{2^{-x}}_{>0} > 0$

$\forall x \in \mathbb{R}$

$$7+2^{-x} = 2^{x+3}$$

$$7+\frac{1}{2^x} = 2^x \cdot 2^3 \quad / \cdot 2^x$$

$$7\cdot 2^x + 1 = 2^{2x}\cdot 8$$

$$8\cdot 2^{2x} - 7\cdot 2^x - 1 = 0$$

$2^x = t$ $\qquad 8t^2 - 7t - 1 = 0$

$$t_{1,2} = \frac{7\pm\sqrt{49+32}}{16} = \frac{7\pm 9}{16} \begin{cases} 1 \\ -\frac{1}{8} \end{cases}$$

$t_1 = 1$

$2^x = 1$

$\boxed{x=0}$

$\cancel{t_2 = -\frac{1}{8}}$

$\cancel{2^x = -\frac{1}{8}}$

1.4 $\log_4(x-2)+\log_{16}(x-2)+\log_2(x-2)=7$

Uslov: $x-2>0$

$\boxed{x>2}$

$x\in(2,\infty)$

$$\log_{2^2}(x-2)+\log_{2^4}(x-2)+\log_2(x-2)=7$$

$$\frac{1}{2}\log_2(x-2)+\frac{1}{4}\log_2(x-2)+\log_2(x-2)=7$$

$$\log_2(x-2)\left(\frac{1}{2}+\frac{1}{4}+1\right)=7$$

$$\log_2(x-2)\,\frac{2+1+4}{4}=7$$

$$\log_2(x-2)\,\frac{\cancel{7}}{4}=\cancel{7}$$

$$\log_2(x-2)=4$$

$$x-2=2^4$$

$$x=2+16$$

$$\boxed{x=18}\in(2,\infty)$$

1.6 $\log_{x^3} 8 - \log_{\frac{1}{x^2}} 2 = 3$

USLOV: $x^3 > 0 \wedge x^3 \neq 1 \wedge \frac{1}{x^2} > 0 \wedge \frac{1}{x^2} \neq 1$

$\boxed{x > 0} \wedge \boxed{x \neq 1}$ $\qquad x \neq 1$

$x \in (0,1) \cup (1, \infty)$

$\frac{1}{3} \log_x 2^3 - \log_{x^{-2}} 2 = 3$

$\frac{1}{\cancel{3}} \cdot \cancel{3} \log_x 2 - \frac{1}{-2} \log_x 2 = 3$

$\log_x 2 + \frac{1}{2} \log_x 2 = 3$

$\frac{3}{2} \log_x 2 = 3$

$\log_x 2 = 2$

$2 = x^2$

$\boxed{x = \sqrt{2}}$ > 0, $\neq 1$

$\cancel{x = -\sqrt{2}}$ < 0

2. Resiti nejednacine:

2.1 $\log_{\frac{1}{4}}(6+2x) > 0$

$\frac{1}{4} < 1$

$6+2x < \left(\frac{1}{4}\right)^0$

$6+2x < 1$

$2x < -5$

$\boxed{x < -\frac{5}{2}}$

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$\log_{4^{-1}}(6+2x) > 0$

$-\log_4(6+2x) > 0$

$\log_4(6+2x) < 0$

$6+2x < 4^0$

$6+2x < 1$

$2x < -5$

$\boxed{x < -\frac{5}{2}}$

USLOVI: $6+2x > 0$

$2x > -6$

$\boxed{x > -3}$

$-3 \quad -\frac{5}{2}$

$\boxed{x \in \left(-3, -\frac{5}{2}\right)}$

2.2 $\log_{\frac{1}{9}}(x^2-4) \geq \log_{\frac{1}{9}}(2x-1)$

$\frac{1}{9} < 1$

$x^2 - 4 \leq 2x - 1$

$x^2 - 2x - 3 \leq 0$

$x^2 - 2x - 3 = 0$

$x_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} \left\langle \begin{matrix} 3 \\ -1 \end{matrix} \right.$

$\boxed{x \in [-1, 3]}$

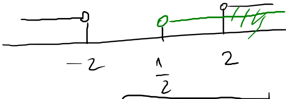
USLOVI:

$x^2 - 4 > 0 \quad \wedge \quad 2x - 1 > 0$

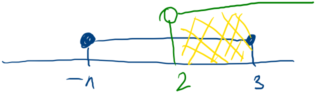
$x^2 = 4 \qquad\qquad 2x > 1$

$x = \pm 2 \qquad\qquad \boxed{x > \frac{1}{2}}$

$\boxed{x \in (-\infty, -2) \cup (2, +\infty)}$



$\boxed{x \in (2, \infty)}$



KONAČNO $\boxed{x \in (2, 3]}$

2.3 $\log_2(\log_4 x) + \log_4(\log_2 x) \leq -4$

UVJET $\boxed{x>0}$ $\wedge$ $\log_4 x > 0$ $\wedge$ $\log_2 x > 0$

$x > 4^0$ $\qquad x > 2^0$

$\boxed{x>1}$ $\qquad x>1$

0 1

$\boxed{x \in (1, +\infty)}$

$\log ab = \log a + \log b$

$$\log_2(\log_{2^2} x) + \log_{2^2}(\log_2 x) \leq -4$$

$$\log_2\left(\tfrac{1}{2}\cdot \log_2 x\right) + \tfrac{1}{2}\log_2(\log_2 x) \leq -4$$

$$\log_2 \tfrac{1}{2} + \log_2(\log_2 x) + \tfrac{1}{2}\log_2(\log_2 x) \leq -4$$

$$\log_2 2^{-1} + \tfrac{3}{2}\log_2(\log_2 x) \leq -4$$

$$-1 \cdot \log_2 2 + \tfrac{3}{2}\log_2(\log_2 x) \leq -4$$

$$-1 + \tfrac{3}{2}\log_2(\log_2 x) \leq -4$$

$$\tfrac{3}{2}\log_2(\log_2 x) \leq -3 \ /\cdot \tfrac{2}{3}$$

$$\log_2(\log_2 x) \leq -2 \qquad 2>1$$

$$\log_2 x \leq 2^{-2}$$

$$\log_2 x \leq \tfrac{1}{4}$$

$$x \leq 2^{\frac{1}{4}}$$

$$\boxed{x \leq \sqrt[4]{2}}$$

1 $\quad \sqrt[4]{2}$

KONAČNO $x \in [1, \sqrt[4]{2}]$

2.4 $\log_{\frac{x}{2}}(x-1) > 2$

$\log_a x > b$

$a > 1 \quad x > a^b$

$0 < a < 1 \quad x < a^b$

USLOVI: $x - 1 > 0 \;\wedge\; \frac{x}{2} > 0 \;\wedge\; \frac{x}{2} \neq 1$

$\boxed{x > 1} \;\wedge\; x > 0 \;\wedge\; \boxed{x \neq 2}$

$\boxed{x \in (1,2) \cup (2, \infty)}$

I $\quad 0 < \frac{x}{2} < 1 \iff \boxed{x < 2}$

$x \in (1,2)$

$x - 1 < \left(\frac{x}{2}\right)^2$

$x - 1 < \frac{x^2}{4} \quad / \cdot 4$

$4x - 4 < x^2$

$-x^2 + 4x - 4 < 0 \quad / \cdot (-1)$

$x^2 - 4x + 4 > 0$

$(x-2)^2 > 0$

$x - 2 \neq 0$

$x \neq 2$, $\forall x \in \mathbb{R} \setminus \{2\}$

$\boxed{x < 2}$

KONAČNO

$\boxed{x \in (1,2)}$

II $\quad \frac{x}{2} > 1 \iff \boxed{x > 2}$

$x \in (2, \infty)$

$x - 1 > \left(\frac{x}{2}\right)^2$

$x - 1 > \frac{x^2}{4} \quad / \cdot 4$

$4x - 4 > x^2$

$-x^2 + 4x - 4 > 0 \quad / (-1)$

$x^2 - 4x + 4 < 0$

$(x-2)^2 < 0$

NIKAD

$x \in \emptyset$

2.5 $\log_{2x}(x^2+1) < 1$

USLOV: $x^2+1>0 \quad \wedge \quad 2x>0 \quad \wedge \quad 2x \neq 1$

$\forall x \in \mathbb{R} \qquad \boxed{x>0} \qquad \boxed{x \neq \frac{1}{2}}$

$$\boxed{x \in \left(0, \tfrac{1}{2}\right) \cup \left(\tfrac{1}{2}, \infty\right)}$$

I $\quad 0 < 2x < 1 \iff 0 < x < \frac{1}{2}$

$\boxed{x \in \left(0, \tfrac{1}{2}\right)}$

$x^2+1 > (2x)^1$

$x^2+1 > 2x$

$x^2-2x+1>0$

$(x-1)^2>0$

$x-1 \neq 0$

$\boxed{x \neq 1}$

$\boxed{x \in \left(0, \tfrac{1}{2}\right)}$

II $\quad 2x > 1 \iff x > \frac{1}{2}$

$\boxed{x \in \left(\tfrac{1}{2}, \infty\right)}$

$x^2+1 < (2x)^1$

$x^2-2x+1<0$

$(x-1)^2<0$

$\perp \qquad x \notin \emptyset$

$\boxed{x \in \emptyset}$

KONAČNO: $\boxed{x \in \left(0, \tfrac{1}{2}\right)}$

DOMEN

3. Odrediti oblast dfinisanosti funkcije $f(x) = \sqrt{\log \frac{x^2-2x+3}{x+1}}$.

$$f(x) = \sqrt{\log \frac{x^2-2x+3}{x+1}}$$

$$\log \frac{x^2-2x+3}{x+1} \geq 0 \quad \wedge \quad \frac{x^2-2x+3}{x+1} > 0 \quad \wedge \quad x+1 \neq 0$$

$$x \neq -1$$

10^0

$$\frac{x^2-2x+3}{x+1} \geq 1$$

$$\frac{x^2-2x+3}{x+1} - 1 \geq 0$$

$$\frac{x^2-2x+3-x-1}{x+1} \geq 0$$

$$\frac{x^2-3x+2}{x+1} \geq 0$$

$$x_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$

$$\frac{(x-1)(x-2)}{x+1} \geq 0$$

	-1		1		2	
$x-1$	$-$		$-$	$+$		$+$
$x-2$	$-$		$-$		$-$	$+$
$x+1$	$-$	0	$+$		$+$	$+$
	$-$		$+$		$-$	$+$

$$\boxed{x \in (-1, 1] \cup [2, +\infty)}$$

$$x_{1,2} = \frac{2 \pm \sqrt{4-12}}{2} \notin \mathbb{R}$$

$$x^2-2x+3 > 0 \quad \forall x \in \mathbb{R}$$

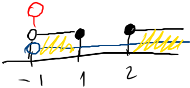
$(+)$

$$\frac{x^2-2x+3}{x+1} > 0$$

$$x+1 > 0$$

$$x > -1$$

$$\boxed{x \in (-1, \infty)}$$



KONAČAN

$$x \in (-1, 1] \cup [2, \infty)$$