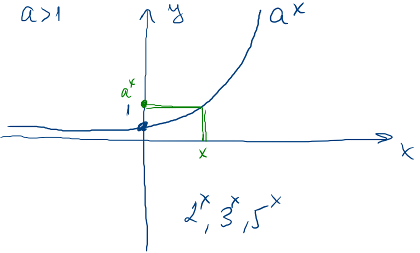
Eksponencijlna funkcija, jednačine i nejednačine

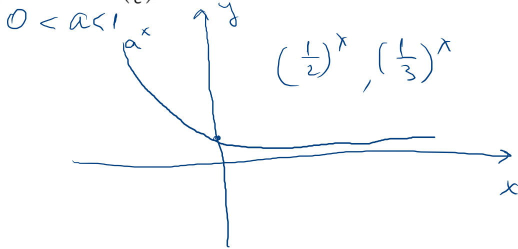
17. април 2024.

Eksponencijalna funkcija

$$y = a^x, \; a > 0, \; a \neq 1,$$

definisana je za svako $x \in \mathbb{R}$ i pozitivna je na celom domenu.
Seče y-osu u tački $(0, 1)$.
Eksponencijalna funkcija je rastuća za $a > 1$, npr. $y = e^x$, a opadajuća za $0 < a < 1$, npr. $y = e^{-x} = \left(\frac{1}{e}\right)^x$.





Eksponencijalna jednačina je oblika

$$a^x = b, \;\; a, b > 0, \;\; a \neq 1.$$

Kako je eksponencijalna funkcija injektivna, važi

$$\boxed{a^x = a^y \quad \Leftrightarrow \quad x = y.}$$

Eksponencijalne nejednačine:

- ako je $a > 1$, onda važi:

$$a^{f(x)} \leq a^{g(x)} \Leftrightarrow f(x) \leq g(x)$$

$a^x \leq a^y \Leftrightarrow x \leq y$

- ako je $0 < a < 1$, onda važi:

$$a^{f(x)} \leq a^{g(x)} \Leftrightarrow f(x) \geq g(x).$$

$2^{x+3} \leq 2^{x^2+1}$

$2^x < 2^3$

$2>1 \quad x<3$

$\left(\frac{1}{2}\right)^x < \left(\frac{1}{2}\right)^3$

$\frac{1}{2}<1 \quad x>3$

Zadaci:

1. U skupu $\mathbb{R}$ rešiti jednačine:

   1.1 $\left(\sqrt{3}\right)^{x^2-x} = 27$

$$\left(3^{\frac{1}{2}}\right)^{x^2-x} = 3^3$$

$$3^{\frac{x^2-x}{2}} = 3^3$$

$$\frac{x^2-x}{2} = 3 \quad / \cdot 2$$

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$x_{1,2} = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm 5}{2} \genfrac{}{}{0pt}{}{3}{-2}$$

$$\boxed{x_1 = 3}, \quad \boxed{x_2 = -2}$$

1.2 $8^x = 7^{x-1} + 7^x$

$$8^x = \underbrace{7^x \cdot 7^{-1}} + \underbrace{7^x}$$

$$8^x = 7^x \left(\frac{1}{7} + 1\right)$$

$$8^x = 7^x \, \frac{8}{7} \qquad \Big| : 7^x > 0$$

$$\frac{8^x}{7^x} = \frac{8}{7}$$

$$\left(\frac{8}{7}\right)^x = \frac{8}{7}$$

$$\boxed{x = 1}$$

$$a^x \cdot a^y = a^{x+y}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

1.3 $3^{x+2}+9^{x+1}=810$

$$3^x \cdot \overset{9}{3^2} + 9^x \cdot 9 = \overset{9 \cdot 90}{810} \quad /: 9$$

$$3^x + (3^2)^x = 90$$

$$3^{2x} + 3^x - 90 = 0$$

$3^x = t$

$t > 0$

$$t^2 + t - 90 = 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1+360}}{2} = \frac{-1 \pm 19}{2} = \begin{cases} 9 \\ -10 \end{cases}$$

$t_1 = 9$

$3^x = 9$

$3^x = 3^2$

$\boxed{x = 2}$

$\cancel{t_2 = -10}$

$\cancel{3^x = -10}$

$$9^{x+1} = (3^2)^{x+1} = 3^{2(x+1)} = 3^{2x+2} = 3^{2x} \cdot 3^2$$

$20 \cdot 20 = 400$

$19 \cdot 19$

171

19

361

1.4 $4^{x^2+2} - 9 \cdot 2^{x^2+2} + 8 = 0$

$$\left(\left(2^2\right)^{x^2+2}\right) - 9 \cdot 2^{x^2+2} + 8 = 0$$

$$\left(2^{x^2+2}\right)^2 - 9 \cdot 2^{x^2+2} + 8 = 0$$

$$2^{x^2+2} = t$$

$$t^2 - 9 \cdot t + 8 = 0$$

$$t_{1,2} = \frac{9 \pm \sqrt{81 - 32}}{2} = \frac{9 \pm 7}{2} = \begin{cases} 8 \\ 1 \end{cases}$$

$$t_1 = 8$$
$$2^{x^2+2} = 8$$
$$2^{x^2+2} = 2^3$$
$$x^2 + 2 = 3$$

$$t_2 = 1$$
$$x^2 = 1$$
$$\boxed{x = \pm 1}$$

$$t_2 = 1$$
$$2^{x^2+2} = 1$$
$$2^{x^2+2} = 2^0$$
$$x^2 + 2 = 0$$
$$\cancel{x^2 = -2}$$

1.5 $9^x + 6^x = 2 \cdot 4^x$

$$3^{2x} + (3\cdot 2)^x = \underbrace{2^1 \cdot 2^{2x}}$$

$$3^{2x} + 3^x \cdot 2^x = 2 \cdot 2^{2x} \quad / : 2^{2x} \neq 0$$

$$\frac{3^{2x}}{2^{2x}} + \frac{3^x \cdot \cancel{2^x}}{2^x \cdot \cancel{2^{2x}}} = 2 \cdot \frac{\cancel{2^{2x}}}{\cancel{2^{2x}}}$$

$$\left(\frac{3}{2}\right)^{2x} + \left(\frac{3}{2}\right)^x = 2$$

$$t^2 + t = 2$$

$$t^2 + t - 2 = 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \left\langle \begin{matrix} 1 \\ -2 \end{matrix} \right.$$

$$\frac{3^x \cdot 2^x}{2^{2x}} = \frac{3^x \cdot \cancel{2^x}}{2^x \cdot \cancel{2^x}}$$

$$\left(\frac{3}{2}\right)^x = t$$

$$2 \cdot 2^{2x} \cancel{=} 4^{2x}$$

$$2^1 \cdot 2^{2x} = 2^{1+2x}$$

$$(ab)^n = a^n \cdot b^n$$

$$t_1 = 1$$

$$\left(\frac{3}{2}\right)^x = 1$$

$$\left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^0$$

$$\boxed{x = 0}$$

$\cancel{t = -2}$

$t > 0$

1.6 $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

$$2^{2x} - 3^x \cdot 3^{-\frac{1}{2}} = 3^x \cdot 3^{\frac{1}{2}} - 2^{2x} \cdot 2^{-1}$$

$$2^{2x} + \frac{1}{2} 2^{2x} = \sqrt{3} \cdot 3^x + \frac{1}{\sqrt{3}} 3^x$$

$$2^{2x}\left(1+\frac{1}{2}\right) = 3^x\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)$$

$$\sqrt{3}$$

$$\sqrt{3}+\frac{1}{\sqrt{3}} = \frac{3+1}{\sqrt{3}}$$

$$\frac{2^{2x}}{3^x} = \left(\frac{2^2}{3}\right)^x$$

$$\frac{3}{2} \cdot 2^{2x} = \frac{4}{\sqrt{3}} \cdot 3^x$$

$$\frac{3}{2}\left(2^2\right)^x = \frac{4}{\sqrt{3}} \cdot 3^x$$

$$\frac{3}{2}\, 4^x = \frac{4}{\sqrt{3}} \cdot 3^x \quad \Big|: 3^x > 0$$

$$\frac{3}{2}\left(\frac{4}{3}\right)^x = \frac{4}{\sqrt{3}}$$

$$\left(\frac{4}{3}\right)^x = \frac{4}{\sqrt{3}} \cdot \frac{2}{3} \qquad 2 = \sqrt{4}$$

$$\left(\frac{4}{3}\right)^x = \frac{4^{1+\frac{1}{2}}}{3^{\frac{1}{2}+1}}$$

$$\left(\frac{4}{3}\right)^x = \left(\frac{4}{3}\right)^{3/2} \rightarrow \boxed{x = \frac{3}{2}}$$

1.7 $2^{x+4} + 2^{x+2} = 5^{x+1} + 3 \cdot 5^x$

$$2^x \cdot 2^4 + 2^x \cdot 2^2 = 5^x \cdot 5 + 3 \cdot 5^x$$

$$2^x (16 + 4) = 8 \cdot 5^x$$

$$2^x \cdot 20 = 8 \cdot 5^x \quad \Big| : 2^x$$

$$\cancel{20}^{\,5} = \cancel{8}^{\,2} \cdot \frac{5^x}{2^x} \quad \Big| : 2$$

$$\frac{5}{2} = \left(\frac{5}{2}\right)^x$$

$$\boxed{x = 1}$$

$$2^x \cdot 2^4 + 2^x \cdot 2^2$$

$$= 2^x \cdot 16 + 2^x \cdot 4$$

$$= 20 \cdot 2^x$$

$$5 \cdot 5^x + 3 \cdot 5^x = 8 \cdot 5^x$$

$$5 \cdot 5^x + 3 \cdot 2^x$$

2. Rešiti nejednačinu:

2.1 $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

$\boxed{\frac{4}{5} < 1}$

$x-1 > 4(1+\sqrt{x})$

$x-1 > 4+4\sqrt{x}$

$x-5 > 4\sqrt{x}$

$4\sqrt{x} < x-5$

$+$

$x-5>0 \wedge x \geq 0 \wedge (4\sqrt{x})^2 < (x-5)^2$

$x>5 \wedge x \geq 0 \wedge 16x < x^2-10x+25$

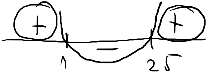
$-x^2+26x-25<0 \quad /\cdot(-1)$

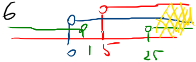
$\boxed{x^2-26x+25>0}$

$x_{1,2} = \frac{26 \pm \sqrt{676-100}}{2} = \frac{26 \pm \sqrt{576}}{2} = \frac{26 \pm 24}{2} <\begin{matrix} 25 \\ 1 \end{matrix}$

$24 \cdot 24$
96
48
576

$26 \cdot 26$
156
52
676





$\boxed{x \in (25, \infty)}$

$x \in (-\infty, 1) \cup (25, +\infty)$

2.2 $2^x + 2^{1-x} - 3 < 0$

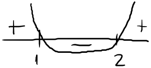
$2^x + 2 \cdot 2^{-x} - 3 < 0$

$2^x + 2 \cdot \frac{1}{2^x} - 3 < 0 \quad / \cdot 2^x > 0$

$2^{2x} + 2 - 3 \cdot 2^x < 0$

$2^x = t \quad , \quad t^2 - 3t + 2 < 0$

$t_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \; \big\langle \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$



$t \in (1, 2)$

$\boxed{a^x > 0 \quad \forall x \in \mathbb{R}}$

$2^x \in (1, 2) \rightarrow 1 < 2^x < 2$

$2^x = 1$

$2^x = 2^0$

$\boxed{x = 0}$

$2^x = 2$

$\boxed{x = 1}$

$2^x > 1$

$2^x > 2^0$

$\boxed{x > 0}$

$2^x < 2$

$2^x < 2^1$

$\boxed{x < 1}$

$\boxed{x \in (0, 1)}$

2.3 $2^{4x+2} \cdot 4^{-x^2} - 3 \cdot 2^{2+2x-x^2} + 8 \leq 0$

$2^{4x} \cdot \underbrace{\boxed{2^2}}_{4} \cdot (2^2)^{-x^2} - 3 \cdot \underbrace{\boxed{2^2}}_{4} \cdot 2^{2x} \cdot 2^{-x^2} + \underbrace{\boxed{8}}_{2\cdot 4} \leq 0 \quad / :4$

$2^{4x} \cdot 2^{-2x^2} - 3 \cdot 2^{2x} \cdot 2^{-x^2} + 2 \leq 0$

$2^{4x-2x^2} - 3 \cdot 2^{2x-x^2} + 2 \leq 0$

$2^{2(2x-x^2)} - 3 \cdot 2^{2x-x^2} + 2 \leq 0$

$2^{2x-x^2} = t \qquad t^2 - 3t + 2 \leq 0$

$t_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \; \begin{cases} 1 \\ 2 \end{cases}$

$t \in (1,2) \longrightarrow 2^{2x-x^2} \in (1,2)$

$1 < t < 2$

$1 < 2^{2x-x^2} < 2$

$1 < 2^{2x-x^2}$

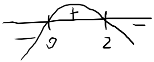
$2^{2x-x^2} > 1$

$2^{2x-x^2} > 2^0$

$2x - x^2 > 0$

$x(2-x) > 0$

$x = 0 \qquad x = 2$



$\boxed{x \in (0,2)}$

$2^{2x-x^2} < 2$

$2^{2x-x^2} < 2^1$

$2x - x^2 < 1$

$-x^2 + 2x - 1 < 0$

$-(x^2 - 2x + 1) < 0$

$-(x-1)^2 < 0$

$x_{1,2} = 1$

$x - 1 \neq 0$

$\boxed{x \neq 1}$

$\boxed{x \in (0,1) \cup (1,2)}$

3. Rešiti sistem jednačina

3.1
$$\begin{aligned} 3^{x+1} &- 2^{y} = \frac{17}{2} \\ 3^{x} &+ 2^{y+1} = 4 \end{aligned}$$

$$3^x \cdot 3 - 2^y = \frac{17}{2}$$
$$3^x + 2^y \cdot 2 = 4$$

$a = 3^x$

$b = 2^y$

$a, b > 0$

$$3a - b = \frac{17}{2} \quad / \cdot 2$$
$$a + 2b = 4$$

$$\left.\begin{aligned} 6a - 2b &= 17 \\ a + 2b &= 4 \end{aligned}\right\} +$$

$$7a = 21$$
$$\boxed{a = 3}$$

$$3 + 2b = 4$$
$$2b = 1$$
$$\boxed{b = \frac{1}{2}}$$

$a = 3^x$

$3 = 3^x$

$\boxed{x = 1}$

$b = 2^y$

$\frac{1}{2} = 2^y$

$2^{-1} = 2^y$

$\boxed{y = -1}$