Vektorski prostori

December 27, 2021

1. U vektorskom prostoru $\boldsymbol{R}^3$ ispitati linearnu zavisnost i generatornost sledećih skupova vektora:

   1.1 $b_1 = (0, 1, 0)$, $b_2 = (0, 0, -1)$

$\dim(\mathbb{R}^3) = 3$

NISU GENERATORNI

JESU LINEARNO NEZAVISNI

1.2 $\boxed{b_1 = (3, 0, 0),\ b_2 = (0, 3, 0), b_3 = (0, 0, 3)},\ b_4 = (5, 7, 9)$

LINEARNO ZAVISNI

$\Rightarrow$ LINEARNO NEZAVISNI I IMA IH 3, A $\dim(\mathbb{R}^3) = 3$

$\Rightarrow$ ČINE BAZU $\Rightarrow$ ONDA SU GENERATORNI

JESU GENERATORNI

1.3 $b_1 = (0, 0, 7)$

$\alpha(0,0,7) = 0$

NIJE GENERATORAN

$\alpha = 0$

LINEARNO NEZAVISAN

1.4 $b_1 = (0,0,0)$, $b_2 = (5,3,1)$

$\alpha b_1 + \beta b_2 = 0$

0

$3 \cdot b_1 + 0 \cdot b_2 = 0$

NISU GENERATORNI

LINEARNO ZAVISNI

I

1.5 $b_1 = (5,5,5)$, $b_2 = (3,3,0)$, $b_3 = (2,0,0)$

JESTE BAZA

$\Rightarrow$ LIN. NEZAVISNI

I GENERATORNI

$(a,0,0)$
$(0,b,0)$
$(0,0,c)$

$a \neq 0$
$b \neq 0$
$c \neq 0$

(a,a,a)
$(b,b,0)$
$(c,0,0)$

II

$\alpha b_1 + \beta b_2 + \gamma b_3 = 0$

$\alpha(5,5,5) + \beta(3,3,0) + \gamma(2,0,0) = 0$

$(5\alpha, 5\alpha, 5\alpha) + (3\beta, 3\beta, 0) + (2\gamma, 0, 0) = 0$

$(5\alpha + 3\beta + 2\gamma,\ 5\alpha + 3\beta,\ 5\alpha) = 0$

$5\alpha = 0 \Rightarrow \alpha = 0$

$5\alpha + 3\beta = 0 \Rightarrow \beta = 0$

$5\alpha + 3\beta + 2\gamma = 0 \Rightarrow \gamma = 0$

(a,a,a)
$(0,b,b)$
$(0,0,c)$

$\Rightarrow$ lin. nezavisni
$\Rightarrow$ baza
$\Rightarrow$ generatori

$\mathbb{R}^4$

— LIN-NEZ.
— GENERATORNE
— BAZA

a) $\{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0)\}$

b) $\{(1,1,1,1),\ (2,2,2,2),\ (3,3,3,3),\ (4,4,4,4)\}$

$(1,2,3,4)$

c) $\{(0,0,0,0),\ (1,2,3,4)\}$

d) $\{(1,1,1,1),\ (1,1,1,0),\ (1,1,0,0),\ (1,0,0,0)\}$

2. Dati su vektori a_1, a_2, a_3 i b. Odrediti realan parametar k tako da se vektor b može izraziti kao linearna kombinacija vektora a_1, a_2 i a_3.

2.1 $a_1 = (4,4,3)$, $a_2 = (7,2,1)$, $a_3 = (4,1,6)$ i $b = (5,9,k)$

$$\boxed{b = \alpha a_1 + \beta a_2 + \gamma a_3}$$

$$(5,9,k) = \alpha(4,4,3) + \beta(7,2,1) + \gamma(4,1,6)$$

$$(5,9,k) = (4\alpha, 4\alpha, 3\alpha) + (7\beta, 2\beta, \beta) + (4\gamma, \gamma, 6\gamma)$$

$$(5,9,k) = (4\alpha + 7\beta + 4\gamma,\ 4\alpha + 2\beta + \gamma,\ 3\alpha + \beta + 6\gamma)$$

$$4\alpha + 7\beta + 4\gamma = 5$$
$$4\alpha + 2\beta + \gamma = 9$$
$$3\alpha + \beta + 6\gamma = k$$

$$\gamma + 4\alpha + 2\beta = 9 \quad \cdot(-4)\ \cdot(-6)$$
$$4\gamma + 4\alpha + 7\beta = 5$$
$$6\gamma + 3\alpha + \beta = k$$

$$\gamma + 4\alpha + 2\beta = 9$$
$$-12\alpha - \beta = -31$$
$$-21\alpha - 11\beta = -56 + k$$

$$\gamma + 2\beta + 4\alpha = 9$$
$$-\beta - 12\alpha = -31 \quad \cdot(-11)$$
$$-11\beta - 21\alpha = k - 56$$

$$\gamma + 2\beta + 4\alpha = 9$$
$$-\beta - 12\alpha = -31$$
$$111\alpha = k + 285$$

$\alpha =$ ____

$\beta =$ ____

$\gamma =$ ____

ODREĐEN

$k \in \mathbb{R}$

$12 \cdot 11$
12
12
132

341
-56

2.2 $a_1 = (2,1,0)$, $a_2 = (-3,2,1)$, $a_3 = (5,-1,-1)$ i $b = (8, k, -2)$

$$* \quad \boxed{b = \alpha a_1 + \beta a_2 + \gamma a_3}$$

$$(8,k,-2) = \alpha(2,1,0) + \beta(-3,2,1) + \gamma(5,-1,-1)$$

$$(8,k,-2) = (2\alpha - 3\beta + 5\gamma,\ \alpha + 2\beta - \gamma,\ \beta - \gamma)$$

$$\begin{aligned} 2\alpha - 3\beta + 5\gamma &= 8 \\ \alpha + 2\beta - \gamma &= k \\ \beta - \gamma &= -2 \end{aligned}$$

$$\begin{aligned} \alpha + 2\beta - \gamma &= k \quad \big) \cdot (-2) \\ 2\alpha - 3\beta + 5\gamma &= 8 \\ \beta - \gamma &= -2 \end{aligned}$$

$$\begin{aligned} \alpha + 2\beta - \gamma &= k \\ -7\beta + 7\gamma &= 8 - 2k \\ \beta - \gamma &= -2 \end{aligned}$$

$$\begin{aligned} \alpha + 2\beta - \gamma &= k \\ \beta - \gamma &= -2 \quad \big) \cdot 7 \\ -7\beta + 7\gamma &= 8 - 2k \end{aligned}$$

$$\begin{aligned} \alpha + 2\beta - \gamma &= k \\ \beta - \gamma &= -2 \\ 0 &= -6 - 2k \end{aligned}$$

$k \neq -3$ ↯

NEMA RJEŠENJA, TJ. $\nexists\ \alpha, \beta, \gamma$ TAKVIH DA JE $*$

$\boxed{k = -3}$

NEODREĐEN

$\exists$ BESKONAČNO MNOGO α, β, γ, TAKVIH DA JE $*$

2.3 $a_1 = (-1, 3, -4)$, $a_2 = (1, -3, 4)$, $a_3 = (2, -6, 8)$ i $b = (0, k, -1)$.

$$b = \alpha a_1 + \beta a_2 + \gamma a_3$$

$$(0, k, -1) = \alpha(-1, 3, -4) + \beta(1, -3, 4) + \gamma(2, -6, 8)$$

$$(0, k, -1) = (-\alpha + \beta + 2\gamma,\ 3\alpha - 3\beta - 6\gamma,\ -4\alpha + 4\beta + 8\gamma)$$

$$\begin{aligned} -\alpha + \beta + 2\gamma &= 0 \quad \cdot 3,\ \cdot(-4) \\ 3\alpha - 3\beta - 6\gamma &= k \\ -4\alpha + 4\beta + 8\gamma &= -1 \end{aligned}
\qquad
\begin{aligned} -\alpha + \beta + 2\gamma &= 0 \\ 0 &= k \\ 0 &= -1 \end{aligned}$$

NEMA REŠENJA

$\Rightarrow \nexists\ \alpha, \beta, \gamma$ TAKVI

DA $b = \alpha a_1 + \beta a_2 + \gamma a_3$

3. Ispitati linearnu zavisnost vektora:

3.1 $(-4, 2, -1, 3)$, $(1, -3, 2, 4)$, $(-2, 4, 3, -1)$, $(-3, 5, 1, -2)$;

DOMAĆI $\rightarrow$ 3.2 $(1, 1, 2, 1)$, $(1, -1, 1, 2)$, $(-3, 1, -4, -5)$, $(0, 2, 1, -1)$.

$$\alpha(-4,2,-1,3)+\beta(1,-3,2,4)+\gamma(-2,4,3,-1)+\delta(-3,5,1,-2)=0$$

$$(-4\alpha+\beta-2\gamma-3\delta,\ 2\alpha-3\beta+4\gamma+5\delta,\ -\alpha+2\beta+3\gamma+\delta,\ 3\alpha+4\beta-\gamma-2\delta)=0$$

$$\begin{aligned}
-4\alpha+\beta-2\gamma-3\delta&=0 \quad -4\\
2\alpha-3\beta+4\gamma+5\delta&=0 \quad \cdot 2\\
-\alpha+2\beta+3\gamma+\delta&=0\\
3\alpha+4\beta-\gamma-2\delta&=0 \quad 3
\end{aligned}$$

$$\begin{aligned}
-\alpha+2\beta+3\gamma+\delta&=0\\
-7\beta-14\gamma-7\delta&=0 \quad /:7\\
\beta+10\gamma+7\delta&=0\\
10\beta+8\gamma+\delta&=0
\end{aligned}$$

$$\begin{aligned}
-\alpha+2\beta+3\gamma+\delta&=0\\
-\beta-2\gamma-\delta&=0\\
\beta+10\gamma+7\delta&=0\\
10\beta+8\gamma+\delta&=0
\end{aligned}$$

$$\begin{aligned}
-\alpha+2\beta+3\gamma+\delta&=0\\
-\beta-2\gamma-\delta&=0\\
8\gamma+6\delta&=0 \quad /:2\\
28\gamma-9\delta&=0
\end{aligned}$$

$$\begin{aligned}
-\alpha+2\beta+3\gamma+\delta&=0\\
-\beta-2\gamma-\delta&=0\\
4\gamma+3\delta&=0\\
28\gamma-9\delta&=0 \quad \cdot(-7)
\end{aligned}$$

$$\begin{aligned}
-\alpha+2\beta+3\gamma+\delta&=0\\
-\beta-2\gamma-\delta&=0\\
4\gamma+3\delta&=0\\
-30\delta&=0
\end{aligned}$$

$$\boxed{\begin{aligned}\delta&=0\\ \gamma&=0\\ \beta&=0\\ \alpha&=0\end{aligned}}$$

$\Rightarrow$ VEKTORI LINEARNO NEZAVISNI

$\mathbb{R}^3$

$\dim(\mathbb{R}^3) = 3$

4. Dati su vektori: $a_1 = (3, 1, 1)$, $a_2 = (m, -1, 0)$ i $a_3 = (0, 1, m)$.

4.1 Za koju vrednost realnog parametra m skup $\{a_1, a_2, a_3\}$ predstavlja bazu prostora $\boldsymbol{R}^3$?

HOMOGEN SISTEM

ODREĐEN: $D_S \neq 0$, IMA SAMO REŠENJE $(0,0,0)$ $\Downarrow$ VEKTORI SU LIN. NEZAVISNI

NEODREĐEN: $D = 0$, ∞-MNOGO REŠENJA $\Downarrow$ VEKTORI LIN. ZAVISNI

$$\alpha a_1 + \beta a_2 + \gamma a_3 = 0$$

$$\alpha(3,1,1) + \beta(m,-1,0) + \gamma(0,1,m) = 0$$

$$(3\alpha + m\beta,\ \alpha - \beta + \gamma,\ \alpha + m\gamma) = 0$$

$$\begin{aligned} 3\alpha + m\beta &= 0 \\ \alpha - \beta + \gamma &= 0 \\ \alpha + m\gamma &= 0 \end{aligned}$$

$D_S \neq 0$ ZA $m \neq 0$ I $m \neq -1$ I TADA JE SISTEM ODREĐEN, PA JE JEDINO NJEGOVO REŠENJE $(\alpha, \beta, \gamma) = (0,0,0)$ ŠTO ZNAČI DA JE SKUP $\{a_1, a_2, a_3\}$ - LINEARNO NEZAVISAN

$$D_S = \begin{vmatrix} 3 & m & 0 \\ 1 & -1 & 1 \\ 1 & 0 & m \end{vmatrix} \begin{matrix} 3 & m \\ 1 & -1 \\ 1 & 0 \end{matrix} = -3m + m - m^2 = -m^2 - 2m = -m(m+1)$$

$\dim(\mathbb{R}^3) = 3$

$\{a_1, a_2, a_3\}$ - LN

$\Downarrow$

$\{a_1, a_2, a_3\}$ - BAZA

4.2 Za $m = 2$ napisati vektor $b = (4, 6, 8)$ kao linearnu kombinaciju vektora a_1, a_2 i a_3.

$a_1 = (3,1,1)$

$a_2 = (2,-1,0)$

$a_3 = (0,1,2)$

$$b = \alpha a_1 + \beta a_2 + \gamma a_3$$

$$(4,6,8) = \alpha(3,1,1) + \beta(2,-1,0) + \gamma(0,1,2)$$

$$(4,6,8) = (3\alpha + 2\beta,\ \alpha - \beta + \gamma,\ \alpha + 2\gamma)$$

$$\begin{aligned} 3\alpha + 2\beta \quad &= 4 \\ \alpha - \beta + \gamma &= 6 \\ \alpha \quad + 2\gamma &= 8 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \quad \cdot(-3),\ \cdot(-1) \\ 3\alpha + 2\beta \quad &= 4 \\ \alpha \quad + 2\gamma &= 8 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ 5\beta - 3\gamma &= -14 \\ \beta + \gamma &= 2 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ \beta + \gamma &= 2 \quad \cdot(-5) \\ 5\beta - 3\gamma &= -14 \end{aligned}$$

$$\begin{aligned} \alpha - \beta + \gamma &= 6 \\ \beta + \gamma &= 2 \\ -8\gamma &= -24 \end{aligned}$$

$\gamma = 3$

$\beta = -1$

$\alpha = 2$

$$b = 2a_1 - a_2 + 3a_3$$

$\mathbb{R}^4$

$\dim(\mathbb{R}^4) = 4$

5. Dokazati da vektori $a = (2, 0, 0, 0)$, $b = (0, -1, 2, 0)$, $c = (0, 0, -3, 0)$, $d = (-1, 0, 0, 1)$ čine bazu vektorskog prostora $\boldsymbol{R}^4$, a zatim napisati vektor $v = (1, 2, -1, 3)$ kao linearnu kombinaciju vektora a, b, c i d.

$$\alpha a + \beta b + \gamma c + \delta d = 0$$

$$\alpha(2,0,0,0) + \beta(0,-1,2,0) + \gamma(0,0,-3,0) + \delta(-1,0,0,1) = 0$$

$$(2\alpha - \delta, -\beta, 2\beta - 3\gamma, \delta) = 0$$

$$\left.\begin{aligned} 2\alpha \quad - \delta &= 0 & \alpha &= 0 \\ -\beta \quad &= 0 \Rightarrow & \boxed{\beta = 0} \\ 2\beta - 3\gamma \quad &= 0 & \gamma &= 0 \\ \delta &= 0 \Rightarrow & \boxed{\delta = 0} \end{aligned}\right\}$$

$\alpha a + \beta b + \gamma c + \delta d = 0$

isključivo kod za $\gamma = \alpha - \beta = \delta = 0$

$\Rightarrow a, b, c, d$ – linearno nezavisni

Kako je $\dim(\mathbb{R}^4) = 4$ i $\{a, b, c, d\}$ su lin. nezavisni oni čine bazu

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$v = (1, 2, -1, 3)$ $\qquad (1, 2, -1, 3) = \alpha a + \beta b + \gamma c + \delta d$

$$\alpha(2,0,0,0)+\beta(0,-1,+2,0)+\gamma(0,0,-3,0)+\delta(-1,0,0,1)=(1,2,-1,3)$$

$$(2\alpha-\delta,\ -\beta,\ 2\beta-3\gamma,\ \delta)=(1,2,-1,3)$$

$$\begin{aligned} 2\alpha-\delta &= 1 \\ -\beta &= 2 \\ 2\beta-3\gamma &= -1 \\ \delta &= 3 \end{aligned}$$

$\delta = 3 \qquad \alpha = 2$

$\beta = -2 \qquad \gamma = -1$

$$\boxed{v = 2a - 2b - c + 3d}$$

6. Skup vektora $A = \{x, y, u, v\}$ čini bazu vektorskog prostora $\boldsymbol{R}^4$. Da li je skup vektora $B = \{x+u, 2y+v, x+u-v, y-3u\}$ baza tog prostora?

$$\alpha(x+u) + \beta(2y+v) + \gamma(x+u-v) + \delta(y-3u) = 0$$

$$(\alpha+\gamma)x + (2\beta+\delta)y + (\alpha+\gamma-3\delta)u + (\beta-\gamma)v = 0$$

$$\Rightarrow \left.\begin{array}{l} \alpha + \gamma = 0 \\ 2\beta + \delta = 0 \\ \alpha + \gamma - 3\delta = 0 \\ \beta - \gamma = 0 \end{array}\right\} \qquad \begin{array}{l} \Rightarrow \alpha = -\gamma \\ \\ \\ \Rightarrow \beta = \gamma \end{array}$$

$$2\gamma + \delta = 0$$
$$-3\delta = 0$$

$$\boxed{\delta = 0}$$
$$\boxed{\gamma = 0}$$
$$\boxed{\alpha = \beta = 0}$$

$\underline{V}$

7. Vektorski prostor V generisan je vektorima $v_1 = (a, 1, 1)$, $v_2 = (-a, a, -a^2)$ i $v_3 = (a^3, -a, 1)$. Naći njegovu dimenziju i bazu u zavisnosti od realnog parametra a.

$\boxed{\dim(V) = \operatorname{rang}(M)}$

$$M = \begin{bmatrix} a & -a & a^3 \\ 1 & a & -a \\ 1 & -a^2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & -a \\ 1 & -a^2 & 1 \\ a & -a & a^3 \end{bmatrix} \begin{matrix} \cdot(-1) \\ \\ \cdot(-a) \end{matrix} \sim \begin{bmatrix} 1 & a & -a \\ 0 & -a^2-a & 1+a \\ 0 & -a^2-a & a^3+a^2 \end{bmatrix} \cdot(-1)$$

$$\sim \begin{bmatrix} 1 & a & -a \\ 0 & -a^2-a & 1+a \\ 0 & 0 & a^3+a^2-a-1 \end{bmatrix} \sim \begin{bmatrix} 1 & a & -a \\ 0 & -a^2-a & 1+a \\ 0 & 0 & (a+1)^2(a-1) \end{bmatrix}$$

$$a^3+a^2-a-1 = a^2(a+1)-(a+1) = (a+1)(a^2-1) = (a+1)^2(a-1)$$

I slučaj:

$-a^2-a \neq 0$, $(a+1)^2(a-1) \neq 0$

$-a(a+1) \neq 0$ $\quad$ $a \neq -1$ $\quad$ $a \neq 1$

$a \neq 0$ $\quad$ $a \neq -1$

$\operatorname{rang}(M) = \dim(V) = 3$

$\{v_1, v_2, v_3\}$ – BAZA

$a \neq 0$, $a \neq -1$, $a \neq 1$

II słucaj: $\boxed{a=0}$

$$M=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \Rightarrow \operatorname{rang}(M)=2 \Rightarrow \dim(V)=2$$

$V_1=(0,1,1)$ $\{V_1, V_3\}$ – BAZA

$V_2=(0,0,0)$

$V_3=(0,0,1)$

III słucaj: $\boxed{a=-1}$

$$M=\begin{bmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \Rightarrow \operatorname{rang}(M)=1 \Rightarrow \dim(V)=1$$

$V_1=(-1,1,1)$ $\{V_1\}$ – BAZA

$V_2=(1,-1,-1)$ $\{V_2\}$ – BAZA

$V_3=(-1,1,1)$ $\{V_3\}$ – BAZA

IV przypadek: $\boxed{a=1}$

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rang}(M) = 2 \Rightarrow \dim(V) = 2$$

$v_1 = (1,1,1)$

$v_2 = (-1,1,-1)$

$v_3 = (1,-1,1)$

$\{v_1, v_2\}$ – BAZA

$\{v_1, v_3\}$ – BAZA

$\{v_2, v_3\}$ – NIE BAZA BO

$v_3 = -v_1$

$\dim(V) = \text{rang}(M)$

8. U zavisnosti od realnog parametra a odrediti bazu i dimenziju prostora S generisanog vektorima $v_1 = (a, a, a, a)$, $v_2 = (a, 2, 2, 2)$, $v_3 = (a, 2, a, a)$ i $v_4 = (a, 2, a, 3)$.

$$M = \begin{bmatrix} a & a & a & a \\ a & 2 & 2 & 2 \\ a & 2 & a & a \\ a & 2 & a & 3 \end{bmatrix} \sim \begin{bmatrix} a & a & a & a \\ 0 & 2-a & 2-a & 2-a \\ 0 & 2-a & 0 & 0 \\ 0 & 2-a & 0 & 3-a \end{bmatrix} \sim \begin{bmatrix} a & a & a & a \\ 0 & 2-a & 2-a & 2-a \\ 0 & 0 & -2+a & -2+a \\ 0 & 0 & -2+a & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} a & a & a & a \\ 0 & 2-a & 2-a & 2-a \\ 0 & 0 & a-2 & a-2 \\ 0 & 0 & 0 & 3-a \end{bmatrix}$$

I slučaj: $a \neq 0$, $2-a \neq 0$, $3-a \neq 0$

$a \neq 0$, $a \neq 2$, $a \neq 3$

$\text{rang}(M) = 4 \Rightarrow \dim(S) = 4$

$\{v_1, v_2, v_3, v_4\}$ - BAZA

II slučaj: $a = 0$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\Rightarrow \text{rang}(M) = 3$

$\Rightarrow \dim(S) = 3$

$v_1 = (0,0,0,0)$

$v_2 = (0,2,2,2)$

$v_3 = (0,2,0,0)$

$v_4 = (0,2,0,3)$

$\{v_2, v_3, v_4\}$ - BAZA

v_1 - NE MOŽE BITI U BAZI JER JE 0 VEKTOR

III słucaj: $\boxed{a=2}$

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rang $(M) = 2 \Rightarrow$ dim $(S) = 2$

$V_1 = (2,2,2,2)$

$V_2 = (2,2,2,2)$

$V_3 = (2,2,2,2)$

$V_4 = (2,2,2,3)$

$\{V_1, V_4\}$ - BAZA

$\{V_2, V_4\}$ - BAZA

$\{V_3, V_4\}$ - BAZA

IV słucaj $\boxed{a=3}$

$$M = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rang $(M) = 3 \Rightarrow$ dim $(S) = 3$

$V_1 = (3,3,3,3)$

$V_2 = (3,2,2,2)$

$V_3 = (3,2,3,3)$

$V_4 = (3,2,3,3)$

$\{V_1, V_2, V_3\}$ BAZA

$\{V_1, V_2, V_4\}$ - BAZA

$$M \cdots \quad N \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 5 & 6 \\ 0 & 0 & a-2 & 0 \end{bmatrix}$$

$a \neq 2$ rang $= 3$ dim 3

$a = 2$ rang 2 dim 2

$$\begin{bmatrix} a & a & a & a \\ 0 & 0 & a-1 & a+2 \\ 0 & 0 & 0 & a+3 \end{bmatrix}$$

$(V, +, \cdot, F)$

$(W, +, \cdot, F)$

$0 \neq W \subseteq V$

9. Ispitati koji od sledećih podskupova $U \subseteq \mathbb{R}^3$ čine (nosače) potprostor prostora $\boldsymbol{R}^3$ ~~i za one koji to jesu odrediti njihovu dimenziju~~:
Napomena: Koristiki sledeće poznate činjenice:
   - Nula vektor vrktorskog prostora mora biti i nula vektor svakog njegovog potprostora. Kako je $0 = (0, 0, 0)$ nula vektor u $\boldsymbol{R}^3$ svi poskupovi koji ne sadrže vektor $0 = (0, 0, 0)$ ne mogu biti potprostori od $\boldsymbol{R}^3$.
   - Da bi U bio potprostor od $\boldsymbol{R}^3$ mora važiti $\forall \alpha, \beta \in \mathbb{R}, \forall u, v \in U, \ \alpha u + \beta v \in U$ (ili posebno $\alpha u \in U$ i $u + v \in U$).
   - Ako je skup U zadat sistemom jednačina čije su promenljive komponente elemenata skupa U, tada je $\boldsymbol{U}$ potprostor prostora $\boldsymbol{R}^3$ akko je taj sistem, sistem linearnih homogenih jednačina. U tom slučaju je je dimenzija potprostora $\boldsymbol{U}$ jednaka stepenu neodređenosti sistema jednačina.

9.1 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$

JESTE

$x - y = 0$

$y - z = 0$

9.2 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 = y^2\}$

NIJE

$$\left.\begin{array}{c} (1,1,0) \\ (-2,2,0) \end{array}\right\} +$$

$$\overline{}$$

$$(-1,3,0)$$

$$(-1)^2 \neq 3^2$$

9.3 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x = 1\}$

NIJE

$0 \in U$

9.4 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

JE STE

SISTEM HOM. LIN. JED.

9.5 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$

NIJE

$0 \in U$

9.6 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\}$

$= \{(0,0,0)\}$

JESTE

9.7 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0,\ y = 0\}$

SISTEM

HOM. LIN.

JED.

$(0, 0, z)$

JESTE

$(0, 0, z_1)$

$(0, 0, z_2)$

$+ \quad (0, 0, z_1 + z_2)$ ✓

$\alpha(0, 0, z_1) = (0, 0, \alpha z_1)$ ✓

9.8 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0,\ xy = 0\}$

$x = 0 \Rightarrow y = 0$

$y = 0 \Rightarrow x = 0$

$(0, 0, z)$

JESTE KAO PRETHODNI

9.9 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z^2 = 0\}$

$$\left.\begin{array}{r} (-1, 0, 1) \\ (-1, 0, -1) \end{array}\right\} +$$

$$\overline{}$$

$$(-2, 0, 0)$$

$$-2 + 0 + 0^2 \neq 0$$

NIJE

9.10 $U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 0\}$

$$(0, y, 0)$$

10. Za skup $\mathcal{R}_S$ rešenja homogenog sistema S linearnih jednačina dokazati da je potprostor vektorskog prostora $\boldsymbol{R}^4$ i odrediti jednu njegovu bazu

$$S: \begin{array}{rcrcrcrcl} x & + & 2y & + & z & - & 2u & = & 0 \\ -2x & - & 5y & + & z & + & 4u & = & 0 \\ x & - & 3y & + & 16z & - & 2u & = & 0 \end{array}$$

$$\begin{aligned} x+2y+z-2u&=0\\ -y+3z&=0\\ -5y+15z&=0 \end{aligned}$$

$$\begin{aligned} x+2y+z-2u&=0\\ -y+3z&=0\\ 0&=0 \quad u \end{aligned}$$

$$\begin{aligned} x+2y&=-z+2u\\ -y&=-3z \end{aligned}$$

2x neodređen

$z=t,\ t\in\mathbb{R}$

$u=m,\ m\in\mathbb{R}$

$y=3t$

$x=2m-7t$

$$R_S=\{(2m-7t,3t,t,m)\mid t,m\in\mathbb{R}\}$$

∞ - MNOGO REŠENJA

$R_S=\{(2m,0,0,m)+(-7t,3t,t,0)\mid t,m\in\mathbb{R}\}$

$=\{m(2,0,0,1)+t(-7,3,1,0)\mid t,m\in\mathbb{R}\}$

SKUP SVIH LINEARNIH KOMBINACIJA VEKTORA

$\{(2,0,0,1),(-7,3,1,0)\}$

$=\mathcal{L}\{(2,0,0,1),(-7,3,1,0)\}$

LINEAL

LINEAL JE UVEK POTPROSTOR V.P. KOJI SMO POSMATRALI

$\Rightarrow R_S$ JE POTPROSTOR $\mathbb{R}^4$

1) $(2,0,0,1), (-7,3,1,0)$ GENERIŠU R_5 JER SE SVI VEKTORI U R_5 MOGU NAPISATI KAO NJIHOVA LINEARNA KOMBINACIJA

2) $(2,0,0,1), (-7,3,1,0) \rightarrow$ ONI SU LIN. NEZAVISNI

$$\Longrightarrow \{(2,0,0,1), (-7,3,1,0)\}$$

BAZA

11. Za skup $\mathcal{R}_S$ rešenja homogenog sistema S linearnih jednačina dokazati da je potprostor vektorskog prostora $\boldsymbol{R}^3$ i odrediti jednu njegovu bazu

$$S: \begin{array}{rcrcrcl} x & + & 2y & - & 3z & = & 0 \\ 3x & + & 4y & - & 2z & = & 0 \\ -5x & - & 2y & - & 13z & = & 0 \end{array} \quad \cdot(-3),\ \cdot 5$$

$$\begin{aligned} x+2y-3z&=0 \\ -2y+7z&=0 \\ 8y-28z&=0 \end{aligned} \quad \cdot 4$$

$$\begin{aligned} x+2y-3z&=0 \\ -2y+7z&=0 \\ 0&=0 \end{aligned}$$

$$\begin{aligned} x+2y-3z&=0 \\ -2y+7z&=0 \end{aligned}$$

$$\begin{aligned} x+2y&=3z \\ -2y&=-7z \end{aligned}$$

1x neodređen

$z=t,\ t\in\mathbb{R}$

$y=\frac{7}{2}t$

$x=-4t$

$R_S=\{(-4t,\frac{7}{2}t,t)\mid t\in\mathbb{R}\}$

$=\{t(-4,\frac{7}{2},1)\mid t\in\mathbb{R}\}$

SVE MOGUĆE LIN. KOMBINACIJE

$=\mathcal{L}\{(-4,\frac{7}{2},1)\}$

PO TEOREMI R_S JE POTPROSTOR OD $\mathbb{R}^3$

$(-4,\frac{7}{2},1)$ - GENERIŠE R_S

$(-4,\frac{7}{2},1)$ - LIN. NEZ

$\{(-4,\frac{7}{2},1)\}$ - BAZA

$$x + y + z = 0$$
$$ax + y + 3z = 0$$
$$x + ay + az = 0$$

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1) DISKUTOVATI PRIRODU REŠENJA SISTEMA U ZAVISNOSTI OD $a \in \mathbb{R}$

2) $a = 3$ REŠITI SISTEM M. M.

3) $a = 3$ POKAZATI DA JE SKUP REŠENJA OVOG SISTEMA V. POTP. OD $\mathbb{R}^3$ I NAĆI NJEGOVU BAZU

ZA VEŽBU IZ SKRIPTE
Zadatak 11.2, 11.3, 11.4, 11.5, 11.6, 11.7, 11.8, 11.9, 11.10, 11.11, 11.12, 11.13, 11.14, 11.15a, 11.16a, 11.17, 11.18