NEODREĐENI INTEGRAL

Za funkciju $f(x)$ definisanu na nekom intervalu I, funkcija $F(x)$ je **primitivna funkcija** na tom intervalu ako je $F(x)$ diferencijabilna (ima izvod) i važi

$$F'(x) = f(x), \quad \forall x \in I.$$

Pr1. Za date funkcije odrediti njihove primitivne funkcije:

$f(x) = \cos x \Longrightarrow F(x) = \sin x \qquad$ jer $(\sin x)' = \cos x$;

$f(x) = e^x \Longrightarrow F(x) = e^x \qquad$ jer $(e^x)' = e^x$;

$f(x) = x^2 \Longrightarrow F(x) = \dfrac{x^3}{3} \qquad$ jer $\left(\dfrac{x^3}{3}\right)' = x^2$.

Treba primetiti da je, recimo, za funkciju $f(x) = \cos x$ osim gore navedene funkcije $F(x) = \sin x$ primitivna funkcija takođe i funkcija $F_1(x) = \sin x + 3$, jer je i $(\sin x + 3)' = \cos x$, kao i funkcija $F_2(x) = \sin x - \frac{1}{2}$, jer je i $\left(\sin x - \frac{1}{2}\right)' = \cos x$ ili bilo koja funkcija oblika $F(x) = \sin x + c$, gde je $c \in \mathbb{R}$ proizvoljna konstanta, jer je u tom slučaju $F'(x) = (\sin x + c)' = f(x)$.

Dakle, primitivna funkcija nije jednoznačno određena. U opštem slučaju, ako je $F(x)$ primitivna funkcija funkcije $f(x)$ na intervalu I tada je i svaka funkcija oblika $F(x) + c$, gde je $c \in \mathbb{R}$ proizvoljna konstanta, takođe primitivna funkcija funkcije $f(x)$, jer je

$$(F(x) + c)' = F'(x) = f(x), \quad c = const.$$

Što znači da za svaku funkciju $f(x)$ postoji beskonačno mnogo primitivnih funkcija.

Kako za dve primitivne funkcije $F_1(x)$ i $F_2(x)$ funkcije $f(x)$ na istom intervalu važi

$$(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = 0,$$

jer je $F_1'(x) = F_2'(x) = f(x)$, sledi da je $F_1(x) - F_2(x) = c$, tj. da se dve primitivne funkcije iste funkcije mogu razlikovati samo za konstantu.

Nekada se to ne primećuje na prvi pogled.

Pr2. Funkcije $F_1(x) = arctg\, x$ i $F_2(x) = arcctg\, \frac{1}{x}$ su obe primitivne funkcije funkcije $f(x) = \frac{1}{1+x^2}$, $x \neq 0$.

Kako je $F_1'(x) = \dfrac{1}{1+x^2} = f(x)$ to je funkcija $F_1(x)$ primitivna funkcija funkcije $f(x)$, a kako je i $F_2'(x) = \dfrac{-1}{1+\frac{1}{x^2}} \cdot \dfrac{-1}{x^2} = \dfrac{1}{1+x^2} = f(x)$ to je i funkcija $F_2(x)$ primitivna funkcije funkcije $f(x)$.

Skup svih primitivnih funkcija funkcije $f(x)$ na nekom intervalu I naziva se **neodređeni integral** funkcije $f(x)$ na tom intervalu i označava se sa

$$\int f(x)dx.$$

Funkcija $f(x)$ naziva se **podintegralna funkcija**, $f(x)dx$ **podintegralni izraz**, $\int$ **znak integrala** a postupak nalaženja neodredjenog integrala se naziva **integracija**.

Ako je $f(x)$ jedna, bilo koja, primitivna funkcija funkcije $f(x)$ na intervalu I onda je

$$\int f(x)dx = F(x) + c.$$

gde je c proizvoljna konstanta koja se naziva **integraciona konstanta**.

Za svaku funkciju postoji primitivna funkcija (neodređeni integral) na intervalu na kom je ona neprekidna. Mi ćemo uvek "rešavati" integral samo na onom intervalu na kom je podintegralna funkcija neprekidna, tj. na onom intervalu na kom postoji neodređeni integral i to nećemo posebno naglašavati nego ćemo ubuduće podrazumevati. Međutim, postoje neke funkcije čiji se neodređeni integrali ne mogu izraziti preko elementarnih funkcija u konačnom obliku pa će oni za nas biti nerešivi. Na primer, takvi su integrali

$$\int e^{-x^2}\,dx\ ,\quad \int \frac{\sin x}{x}\,dx\ ,\quad \int \frac{dx}{\ln x}\ ,\quad \int \cos x^2\,dx\ ,\quad \int \frac{e^x}{x}\,dx.$$

Osobine neodređenog integrala

Osnovne osovine neodređenog integrala su:

1. $\left(\int f(x)\,dx\right)' = f(x);$

2. $\int F'(x)\,dx = F(x) + c;$

3. $\int \alpha f(x)\,dx = \alpha \int f(x)\,dx, \quad \alpha \in \mathbb{R};$

4. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$

Tablica integrala

Tablica osnovnih integrala se dobija na osnovu tablice izvoda ili neposrednom proverom:

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1,$

2. $\int \frac{dx}{x} = \ln|x| + c, \qquad x \neq 0,$

3. $\int \sin x\,dx = -\cos x + c,$

4. $\int \cos x\,dx = \sin x + c,$

5. $\int \frac{dx}{\cos^2 x} = \text{tg}\, x + c, \qquad x \neq \frac{\pi}{2} + k\pi, \quad k \in Z,$

6. $\int \frac{dx}{\sin^2 x} = -\text{ctg}\, x + c, \qquad x \neq k\pi, \quad k \in Z,$

7. $\int e^x\,dx = e^x + c,$

8. $\int a^x\,dx = \frac{a^x}{\ln a} + c, \qquad a > 0, \quad a \neq 1,$

9. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\text{arctg}\,\frac{x}{a} + c = -\frac{1}{a}\text{arcctg}\,\frac{x}{a} + c_1, \qquad a \neq 0,$

10. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c = -\arccos\frac{x}{a} + c_1, \qquad |x| \leq a,$

11. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + c, \qquad a \in R.$

Zadatak 1. Izračunati integrale:

1. $\int 3\,dx = 3\int dx = 3x + c;$

2. $\int x^5\,dx = \frac{x^{5+1}}{5+1} + c + \frac{x^6}{6} + c;$

3. $\int 10x^4\,dx = 10\int x^4\,dx = 10\cdot\frac{x^{4+1}}{4+1}+c = 2x^5+c$;

4.
$$\begin{aligned}\int\left(3x^2+2x-5\right)\,dx &= \int 3x^2\,dx+\int 2x\,dx-\int 5\,dx = 3\int x^2\,dx+2\int x\,dx-5\int dx = 3\frac{x^{2+1}}{2+1}+2\frac{x^{1+1}}{1+1}-5x+c\\ &= 3\frac{x^3}{3}+2\frac{x^2}{2}-5x+c = x^3+x^2-5x+c;\end{aligned}$$

5. $\int\frac{1}{x^7}\,dx = \int x^{-7}\,dx = \frac{x^{-7+1}}{-7+1}+c+\frac{x^{-6}}{-6}+c = -\frac{1}{6x^6}+c;$

6. $\int\frac{2}{x^5}\,dx = 2\int x^{-5}\,dx = 2\frac{x^{-5+1}}{-5+1}+c = 2\frac{x^{-4}}{-4}+c = -\frac{1}{2x^4}+c$

7.
$$\begin{aligned}\int\left(\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}+33\right)\,dx &= \int\frac{1}{x^3}\,dx+\int\frac{1}{x^2}\,dx-\int\frac{1}{x}\,dx+\int 33\,dx = \int x^{-3}\,dx+\int x^{-2}\,dx-\int\frac{1}{x}\,dx+33\int dx\\ &= \frac{x^{-3+1}}{-3+1}+\frac{x^{-2+1}}{-2+1}-\ln|x|+33x+c = \frac{x^{-2}}{-2}+\frac{x^{-1}}{-1}-\ln|x|+33x+c\\ &= -\frac{1}{2x^2}-\frac{1}{x}-\ln|x|+33x+c;\end{aligned}$$

8. $\int\sqrt{x}\,dx = \int x^{\frac12}\,dx = \frac{x^{\frac12+1}}{\frac12+1} = \frac{x^{\frac32}}{\frac32} = \frac{2\sqrt{x^3}}{3}+c$;

9. $\int\frac{1}{\sqrt{x}}\,dx = \int x^{-\frac12}\,dx = \frac{x^{-\frac12+1}}{-\frac12+1}+c = \frac{x^{\frac12}}{\frac12}+c = 2\sqrt{x}+c$;

10. $\int\frac{1}{\sqrt[3]{x}}\,dx = \int x^{-\frac13}\,dx = \frac{x^{-\frac13+1}}{-\frac13+1}+c = \frac{x^{\frac23}}{\frac23}+c = \frac32\sqrt[3]{x^2}+c;$

11.
$$\begin{aligned}\int\left(\sqrt{x^3}-\sqrt[3]{x^2}+\frac{1}{\sqrt[4]{x}}\right)\,dx &= \int\left(\sqrt{x^3}-\sqrt[3]{x^2}+\frac{1}{\sqrt[4]{x}}\right)\,dx = \int\sqrt{x^3}\,dx-\int\sqrt[3]{x^2}\,dx+\int\frac{1}{\sqrt[4]{x}}\,dx\\ &= \int x^{\frac32}\,dx-\int x^{\frac23}\,dx+\int x^{-\frac14}\,dx = \frac{x^{\frac32+1}}{\frac32+1}-\frac{x^{\frac23+1}}{\frac23+1}+\frac{x^{-\frac14+1}}{-\frac14+1}+c\\ &= \frac{x^{\frac52}}{\frac52}-\frac{x^{\frac53}}{\frac53}+\frac{x^{\frac34}}{\frac34}+c = \frac{2\sqrt{x^5}}{5}-\frac{3\sqrt[3]{x^5}}{5}+\frac{4\sqrt[4]{x^3}}{3}+c;\end{aligned}$$

12.
$$\begin{aligned}\int\left(6x^2+8x+3+\sqrt{x}-\frac{1}{x^3}+\frac{1}{\sqrt[4]{x^3}}\right)\,dx &= 6\int x^2\,dx+8\int x\,dx+3\int dx+\int x^{\frac12}\,dx-\int x^{-3}\,dx+\int x^{-\frac34}\,dx\\ &= 6\frac{x^3}{3}+8\frac{x^2}{2}+3x+\frac{x^{\frac32}}{\frac32}-\frac{x^{-2}}{-2}+\frac{x^{\frac14}}{\frac14}+c\\ &= 2x^3+4x^2+3x+\frac23\sqrt{x^3}+\frac{1}{2x^2}+4\sqrt[4]{x}+c;\end{aligned}$$

13. $\int\left(\frac{12}{x}+7e^x-7^x\right)\,dx = \int\frac{12}{x}\,dx+\int 7e^x\,dx-\int 7^x\,dx = 12\int\frac{1}{x}\,dx+7\int e^x\,dx-\int 7^x\,dx = 12\ln|x|+7e^x-\frac{7^x}{\ln 7}+c;$

14.
$$\begin{aligned}\int\left(x^3+2\sin x-\frac{1}{\sin^2 x}+\frac{1}{1+x^2}\right)\,dx &= \int x^3\,dx+2\int\sin x\,dx-\int\frac{1}{\sin^2 x}\,dx+\int\frac{1}{1+x^2}\,dx\\ &= \frac{x^4}{4}-2\cos x+\operatorname{ctg}\,x+\operatorname{arctg}\,x+c;\end{aligned}$$

15. $\int\frac{dx}{9+x^2} = \int\frac{dx}{3^2+x^2} = \frac13\operatorname{arctg}\frac{x}{3}+c;$

16. $\displaystyle\int \frac{dx}{5+x^2} = \int \frac{dx}{\left(\sqrt{5}\right)^2 + x^2} = \frac{1}{\sqrt{5}}\text{arctg}\ \frac{x}{\sqrt{5}} + c;$

17. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2} + c;$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2+3}} = \ln|x+\sqrt{x^2+3}| + c;$

19. $\displaystyle\int \frac{dx}{\sqrt{x^2-5}} = \ln|x+\sqrt{x^2-5}| + c;$

20.
$$\begin{aligned}\int \left(2^x + \frac{1}{x} + \frac{1}{\sqrt{x^2+4}} - 7e^x + \frac{1}{\sqrt{9-x^2}}\right)\ dx &= \int 2^x\ dx + \int \frac{1}{x}\ dx + \int \frac{1}{\sqrt{x^2+4}}\ dx - 7\int e^x\ dx + \int \frac{1}{\sqrt{9-x^2}}\ dx \\ &= \frac{2^x}{\ln 2} + \ln|x| + \ln|x+\sqrt{x^2+4}| - 7e^x + \arcsin\frac{x}{3} + c;\end{aligned}$$

21. $\displaystyle\int \frac{x^2}{x^2+1}\ dx = \int \frac{x^2+1-1}{x^2+1}\ dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)\ dx = \int dx - \int \frac{dx}{x^2+1} = x - \text{arctg}\ x + c;$

22. $\displaystyle\int \text{tg}\ ^2 x\ dx = \int \frac{\sin^2 x}{\cos^2 x}\ dx = \int \frac{1-\cos^2 x}{\cos^2 x}\ dx = \int \frac{dx}{\cos^2 x} - \int dx = \text{tg}\ x - x + c;$

23. $\displaystyle\int e^x\left(1 - \frac{e^{-x}}{x^2}\right)\ dx = \int \left(e^x - \frac{1}{x^2}\right)\ dx = \int e^x\ dx - \int x^{-2}\ dx = e^x - \frac{x^{-1}}{-1} + c = e^x + \frac{1}{x} + c.$

Smena promenljive

Integrale koji se ne nalaze u tablici potrebno je, pri rešavanju, svesti na tablične. Jedan od dva osnovna metoda kojima se to postiže je integracija pomoću smene.

Neka je $\varphi(t)$ funkcija koja na nekom intervalu realne ose ima neprekidan prvi izvod i neka je na tom intervalu $\varphi'(t) \neq 0$. Tada, ako je $x = \varphi(t)$, na posmatranom intervalu važi

$$\int f(x)\ dx = \int f(\varphi(t))\varphi'(t)\ dt.$$

Zadatak 2. Smenom promenljivih rešiti integral

1. $\displaystyle\int (2x+3)^3\ dx = \begin{pmatrix} 2x+3=t \\ 2\ dx = dt \\ dx = \frac{1}{2}dt \end{pmatrix} = \frac{1}{2}\int t^3\ dt = \frac{1}{2}\cdot\frac{t^4}{4} + c = \frac{(2x+3)^4}{8} + c\ ;$

2. $\displaystyle\int \frac{1}{x+1}\ dx = \begin{pmatrix} x+1=t \\ dx = dt \end{pmatrix} = \int \frac{1}{t}\ dt = \ln|t| + c = \ln|x+1| + c\ ;$

3. $\displaystyle\int \frac{1}{(x+1)^2}\ dx = \begin{pmatrix} x+1=t \\ dx = dt \end{pmatrix} = \int \frac{1}{t^2}\ dt = \int t^{-2}\ dt = \frac{t^{-1}}{-1} + c = -\frac{1}{t} + c = -\frac{1}{x+1} + c\ ;$

4. $\displaystyle\int \sqrt{3x-5}\ dx = \begin{pmatrix} 3x-5=t \\ 3\ dx = dt \\ dx = \frac{1}{3}dt \end{pmatrix} = \int \frac{1}{3}\sqrt{t}\ dt = \frac{1}{3}\int t^{\frac{1}{2}}\ dt = \frac{1}{3}\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{9}\sqrt{t^3} + c = \frac{2}{9}\sqrt{(3x-5)^3} + c\ ;$

5. $\displaystyle\int \frac{1}{\sqrt[3]{x+2}}\ dx = \begin{pmatrix} x+2=t \\ dx = dt \end{pmatrix} = \int \frac{1}{\sqrt[3]{t}}\ dt = \int t^{-\frac{1}{3}}\ dt = \frac{t^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c = \frac{t^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3\sqrt[3]{t^2}}{2} + c = \frac{3\sqrt[3]{(x+2)^2}}{2} + c\ ;$

6. $\displaystyle\int \frac{x}{\sqrt{x^4+1}}\,dx = \begin{pmatrix} x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{pmatrix} = \frac{1}{2}\int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{2}\ln|t+\sqrt{t^2+1}+c = \tfrac{1}{2}\ln|x^2+\sqrt{x^4+1}+c;$

7. $\displaystyle\int x^2\sqrt{x^3+3}\,dx = \begin{pmatrix} x^3+3=t \\ 3x^2\,dx=dt \\ x^2\,dx=\frac{1}{3}dt \end{pmatrix} = \int \frac{1}{3}\sqrt{t}\,dt = \frac{1}{3}\int t^{\frac{1}{2}}\,dt = \frac{1}{3}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{2}{9}\sqrt{t^3}+c = \frac{2}{9}\sqrt{(x^3+3)^3}+c\ ;$

8. $$\begin{aligned}\int\sqrt{x^2+1}x^5\,dx &= \int\sqrt{x^2+1}\,x^4\,x\,dx = \begin{pmatrix} x^2+1=t \Longrightarrow x^2=t-1 \\ 2xdx=dt \Longrightarrow xdx=\frac{1}{2}dt \end{pmatrix} = \int\frac{1}{2}\sqrt{t}\,(t-1)^2\,dt = \frac{1}{2}\int t^{\frac{1}{2}}(t^2-2t+1)\,dt \\ &= \frac{1}{2}\int(t^{\frac{5}{2}}-2t^{\frac{3}{2}}+t^{\frac{1}{2}})\,dt = \frac{1}{2}\Big(\frac{2}{7}t^{\frac{7}{2}}-\frac{4}{5}t^{\frac{5}{2}}+\frac{2}{3}t^{\frac{3}{2}}\Big)+c = \frac{1}{7}\left(x^2+1\right)^{\frac{7}{2}}-\frac{2}{5}\left(x^2+1\right)^{\frac{5}{2}}+\frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}+c \\ &= \frac{1}{7}\left(\sqrt{x^2+1}\right)^7-\frac{2}{5}\left(\sqrt{x^2+1}\right)^5+\frac{1}{3}\left(\sqrt{x^2+1}\right)^3+c;\end{aligned}$$

9. $\displaystyle\int\frac{6x+1}{3x^2+x+1}\,dx = \begin{pmatrix} 3x^2+x+1=t \\ (6x+1)\,dx=dt \end{pmatrix} = \int\frac{1}{t}\,dt = \ln|t|+c = \ln|3x^2+x+1|+c\ ;$

10. $$\begin{aligned}\int\frac{x}{x+2}\,dx &= \int\frac{x+2-2}{x+2}\,dx = \int\frac{x+2}{x+2}\,dx+\int\frac{-2}{x+2}\,dx = \int dx-2\int\frac{1}{x+2}\,dx = \begin{pmatrix} x+2=t \\ dx=dt \end{pmatrix} \\ &= \int dx-2\int\frac{1}{t}\,dt = x-2\ln|t|+c = x-2\ln|x+2|+c;\end{aligned}$$

11. $$\begin{aligned}\int\frac{x}{(x-3)^2}\,dx &= \int\frac{x-3+3}{(x-3)^2}\,dx = \int\frac{x-3}{(x-3)^2}\,dx+\int\frac{3}{(x-3)^2}\,dx = \int\frac{1}{x-3}\,dx+3\int\frac{1}{(x-3)^2}\,dx = \begin{pmatrix} x-3=t \\ dx=dt \end{pmatrix} \\ &= \int\frac{1}{t}\,dt+3\int\frac{1}{t^2}\,dt = \ln|t|-3\tfrac{1}{t}+c = \ln|x-3|-\tfrac{3}{x-3}+c;\end{aligned}$$

12. $\displaystyle\int e^{2x}\,dx = \begin{pmatrix} 2x=t \\ 2\,dx=dt \end{pmatrix} = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t+c = \frac{1}{2}e^{2x}+c\ ;$

13. $\displaystyle\int\sqrt[3]{e^x}\,dx = \int e^{\frac{x}{3}}\,dx = \begin{pmatrix} \frac{x}{3}=t \\ \frac{1}{3}\,dx=dt \end{pmatrix} = 3\int e^t\,dt = 3e^t+c = 3e^{\frac{x}{3}}+c = 3\sqrt[3]{e^x}+c\ ;$

14. $\displaystyle\int\frac{5}{\sqrt{e^{5x}}}\,dx = 5\int e^{-\frac{5x}{2}}\,dx = \begin{pmatrix} -\frac{5x}{2}=t \\ -\frac{5}{2}\,dx=dt \end{pmatrix} = 5\cdot\Big(-\frac{2}{5}\Big)\int e^t\,dt = -2e^t+c = -2e^{-\frac{5x}{2}}+c = -\frac{2}{\sqrt{e^{5x}}}+c\ ;$

15. $\displaystyle\int 2xe^{x^2}\,dx = \begin{pmatrix} x^2=t \\ 2xdx=dt \end{pmatrix} = \int e^t\,dt = e^t+c = e^{x^2}+c\ ;$

16. $\displaystyle\int\frac{e^{2x}}{1+e^x}\,dx = \int\frac{e^x\cdot e^x}{1+e^x}\,dx = \begin{pmatrix} 1+e^x=t \\ e^x=t-1 \\ e^x\,dx=dt \end{pmatrix} = \int\frac{t-1}{t}\,dt = \int dt-\int\frac{1}{t}\,dt = t-\ln|t|+c = 1+e^x-\ln e^x+c;$

17. $\displaystyle\int e^x\sqrt{1+e^x}\,dx = \begin{pmatrix} 1+e^x=t \\ e^x\,dx=dt \end{pmatrix} = \int\sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}}+c = \frac{2}{3}\sqrt{(1+e^x)^3}+c\ ;$

18. $\displaystyle\int 2^{-4x}\,dx = \begin{pmatrix} -4x=t \\ -4\,dx=dt \end{pmatrix} = -\frac{1}{4}\int 2^t\,dt = -\frac{1}{4}\cdot\frac{2^t}{\ln 2}+c = -\frac{2^{-4x}}{4\ln 2}+c\ ;$

19. $\displaystyle\int e^{-x^2}x\,dx = \begin{pmatrix} -x^2=t \\ -2x\,dx=dt \\ x\,dx=-\frac{1}{2}dt \end{pmatrix} = -\frac{1}{2}\int e^t\,dt = -\frac{1}{2}e^t+c = -\frac{1}{2}e^{-x^2}+c\ ;$

20. $$\int \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx = \begin{pmatrix} \sqrt[3]{x}=t \\ \frac{1}{3}x^{-\frac{2}{3}}\,dx = dt \\ \frac{1}{\sqrt[3]{x^2}}\,dx = 3dt \end{pmatrix} = 3\int e^t\,dt = 3e^t + c = 3e^{\sqrt[3]{x}} + c\ ;$$

21. $$\int \sin\frac{x}{2}\,dx = \begin{pmatrix} \frac{x}{2}=t \\ \frac{1}{2}\,dx = dt \end{pmatrix} = 2\int \sin t\,dt = -2\cos t + c = -2\cos\frac{x}{2} + c;$$

22. $$\int \cos 5x\,dx = \begin{pmatrix} 5x=t \\ 5\,dx = dt \end{pmatrix} = \frac{1}{5}\int \cos t\,dt = \frac{1}{5}\sin t + c = \frac{1}{5}\sin 5x + c;$$

23. $$\int \sin^5 x\cos x\,dx = \begin{pmatrix} \sin x=t \\ \cos\ dx = dt \end{pmatrix} = \int t^5\,dt = \frac{t^6}{6} + c = \frac{1}{6}\sin^6 x + c;$$

24. $$\int \mathrm{tg}\ x\,dx = \int \frac{\sin x}{\cos x}\,dx = \begin{pmatrix} \cos x=t \\ -\sin x\,dx = dt \end{pmatrix} = -\int \frac{1}{t}\,dt = -\ln|t| + c = -\ln|\cos x| + c;$$

25. $$\begin{aligned}\int \frac{\mathrm{arctg}\ \frac{x}{2}}{4+x^2}\,dx &= \frac{1}{4}\int \frac{\mathrm{arctg}\ \frac{x}{2}}{1+\left(\frac{x}{2}\right)^2}\,dx = \begin{pmatrix} \frac{x}{2}=t \\ \frac{1}{2}\,dx = dt \end{pmatrix} = \frac{1}{4}\int \frac{\mathrm{arctg}\ t}{1+t^2}\,dx = \begin{pmatrix} \mathrm{arctg}\ t = m \\ \frac{dt}{1+t^2} = dm \end{pmatrix} = \frac{1}{4}\int m\,dm \\ &= \frac{1}{4}\frac{m^2}{2} + c = \frac{1}{8}\mathrm{arctg}\ ^2 t + c = \frac{1}{8}\mathrm{arctg}\ ^2\frac{x}{2} + c;\end{aligned}$$

26. $$\int \frac{\mathrm{arctg}^2 x}{1+x^2}\,dx = \begin{pmatrix} \mathrm{arctg}\ x=t \\ \frac{1}{1+x^2}\,dx = dt \end{pmatrix} = \int t^2\,dt = \frac{t^3}{3} + c = \frac{1}{3}\mathrm{arctg}\ ^3 x + c;$$

27. $$\int \frac{\sqrt{\mathrm{arctg}\ 2x}}{1+4x^2}\,dx = \begin{pmatrix} \mathrm{arctg}\ 2x=t \\ \frac{2}{1+4x^2}\,dx = dt \end{pmatrix} = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\frac{2}{3}t^{\frac{3}{2}} + c = \frac{1}{3}\sqrt{\mathrm{arctg}\ ^2 2x} + c;$$

28. $$\int \frac{e^{\mathrm{arctg}\ x}}{1+x^2}\,dx = \begin{pmatrix} \mathrm{arctg}\ x=t \\ \frac{1}{1+x^2}\,dx = dt \end{pmatrix} = \int e^t\,dt = e^t + c = e^{\mathrm{arctg}\ x} + c;$$

29. $$\int \frac{x\ln\left(1+x^2\right)}{1+x^2}\,dx = \begin{pmatrix} \ln\left(1+x^2\right)=t \\ \frac{2x}{1+x^2}\,dx = dt \end{pmatrix} = \frac{1}{2}\int t\,dt = \frac{1}{2}\frac{t^2}{2} + c = \frac{1}{4}\ln^2\left(1+x^2\right) + c;$$

30. $$\begin{aligned}\int \frac{\ln x}{x\sqrt{1+\ln x}}\,dx &= \begin{pmatrix} 1+\ln x = t^2 \\ \ln x = t^2 - 1 \\ \frac{1}{x}\,dx = 2t\,dt \end{pmatrix} = 2\int \frac{t^2-1}{t}t\,dt = 2\int t^2\,dt - 2\int dt = 2\frac{t^3}{3} - 2t + c \\ &= \frac{2}{3}\left(\sqrt{1+\ln x}\right)^3 - 2\sqrt{1+\ln x} + c;\end{aligned}$$

31. $$\int \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx = \begin{pmatrix} \ln\left(x+\sqrt{1+x^2}\right) = t \\ \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}}2x\right)\,dx = dt \\ \frac{1}{x+\sqrt{1+x^2}}\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\,dx = dt \\ \frac{1}{\sqrt{1+x^2}}\,dx = dt \end{pmatrix} = \int t\,dt = \frac{t^2}{2} + c = \frac{1}{2}\ln^2\left(x+\sqrt{1+x^2}\right) + c.$$

Parcijalna integracija

Parcijalna integracija je drugi osnovni metod za svođenje netabličnih integrala na tablične.

Neka su funkcije $u(x)$ i $v(x)$ diferencijabilne funkcije na nekom intervalu I. Tada na posmatranom intervalu važi formula za parcijalnu integraciju

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.$$

Ako se uvedu skraćene oznake $u = u(x)$, $v = v(x)$, $du = u'(x)dx$ i $dv = v'(x)dx$ prethodna formula se može napisati kao

$$\int udv = uv - \int vdu.$$

Zadatak 3. Parcijalnom integracijom rešiti integrale

1. $\int \ln x \, dx = \begin{pmatrix} u = \ln x & , & dv = dx \\ du = \frac{dx}{x} & , & v = x \end{pmatrix} = x \ln x - \int dx = x(\ln x - 1) + c;$

2. $$\int \ln^2 x \, dx = \begin{pmatrix} u = \ln^2 x & dv = dx \\ du = 2\frac{1}{x} \ln x \, dx & v = \int dx = x \end{pmatrix} = x \ln^2 x - 2 \int x \cdot \frac{1}{x} \ln x \, dx = x \ln^2 x - 2 \int \ln x \, dx$$
$$= \begin{pmatrix} u = \ln x & dv = dx \\ du = \frac{1}{x} \, dx & v = \int dx = x \end{pmatrix} = x \ln^2 x - 2 (x \ln x - x) + c = x \ln^2 x - 2x \ln x + 2x + c;$$

3. $$\int x e^x \, dx = \begin{pmatrix} u = x & dv = e^x dx \\ du = dx & v = \int e^x dx = e^x \end{pmatrix} = x e^x - \int e^x \, dx = x e^x - e^x + c \ ;$$

4. $$\int (x^2 - 1) e^{2x} \, dx = \begin{pmatrix} u = x^2 - 1 & dv = e^{2x} dx \\ du = 2x dx & v = \int e^{2x} \, dx = \frac{1}{2} e^{2x} \end{pmatrix} = \frac{1}{2}(x^2 - 1)e^{2x} - \frac{1}{2} \cdot 2 \int x e^{2x} \, dx$$
$$= \frac{1}{2}(x^2 - 1)e^{2x} - \int x e^{2x} \, dx = \begin{pmatrix} u_1 = x & dv_1 = e^{2x} dx \\ du_1 = dx & v_1 = \int e^{2x} \, dx = \frac{1}{2} e^{2x} \end{pmatrix}$$
$$= \frac{1}{2}(x^2 - 1)e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{2} \int e^{2x} \, dx = \frac{1}{2}(x^2 - x - 1)e^{2x} + \frac{1}{4} e^{2x} + c;$$

5. $$\int (x+1)^2 \cos x \, dx = \begin{pmatrix} u = (x+1)^2 & dv = \cos x \, dx \\ du = 2 (x+1) \ dx & v = \int \cos x \, dx = \sin x \end{pmatrix} = (x+1)^2 \sin x - 2 \int (x+1) \sin x \, dx$$
$$= \begin{pmatrix} u = x + 1 & dv = \sin x \, dx \\ du = dx & v = \int \sin x \, dx = -\cos x \end{pmatrix} = (x+1)^2 \sin x - 2 \left(-(x+1) \cos x - \int (-\cos x) \ dx \right)$$
$$= (x+1)^2 \sin x + 2 (x+1) \cos x - 2 \sin x + c;$$

6. $$\int e^x \cos x \, dx = \begin{pmatrix} u = e^x & dv = \cos x dx \\ du = e^x \, dx & v = \int \cos x \, dx = \sin x \end{pmatrix} = e^x \sin x - \int e^x \sin x \, dx$$
$$= \begin{pmatrix} u_1 = e^x & dv_1 = \sin x dx \\ du_1 = e^x \, dx & v_1 = \int \sin x \, dx = -\cos x \end{pmatrix} = e^x \sin x - \left(-e^x \cos x - \int (-e^x \cos x) \ dx \right)$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx;$$

Ovde smo sad dobili integral isti kao onaj od kog smo i krenuli, pa možemo integral prebaciti na levu stranu i tako ga izračunati.
$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$, tj. $\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c;$

7. $$\int x^5 e^{-x^2} \, dx = \int x^4 e^{-x^2} x \, dx = \begin{pmatrix} -x^2 = t \\ -2x \, dx = dt \end{pmatrix} = -\frac{1}{2} \int t^2 e^t dt = \begin{pmatrix} u = t^2 & dv = e^t dt \\ du = 2t \, dt & v = \int e^t \, dt = e^t \end{pmatrix}$$
$$= -\frac{1}{2} \left(t^2 e^t - 2 \int t e^t \, dt \right) = \begin{pmatrix} u_1 = t & dv_1 = e^t \, dt \\ du_1 = dt & v_1 = \int e^t \, dt = e^t \end{pmatrix} = -\frac{1}{2} t^2 e^t + t e^t - \int e^t \, dt$$
$$= -\frac{1}{2} t^2 e^t + t e^t - \int e^t \, dt = -\frac{1}{2} t^2 e^t + t e^t - e^t + c = -\frac{1}{2} x^4 e^{-x^2} + x^2 e^{-x^2} - e^{x^2} + c;$$

8. $$\int x^2 \ln x \, dx = \begin{pmatrix} u = \ln x & dv = x^2 dx \\ du = \frac{1}{x} \, dx & v = \int x^2 \, dx = \frac{x^3}{3} \end{pmatrix} = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + c \ ;$$

9. $$\int (x^3+x+1)\ln x\,dx = \begin{pmatrix} u=\ln x & dv=(x^3+x+1)\,dx \\ du=\frac{1}{x}\,dx & v=\int(x^3+x+1)\,dx \\ & =\frac{x^4}{4}+\frac{x^2}{2}+x \end{pmatrix} = \Big(\frac{x^4}{4}+\frac{x^2}{2}+x\Big)\ln x - \int \Big(\frac{x^4}{4}+\frac{x^2}{2}+x\Big)\frac{1}{x}\,dx$$

$$= \Big(\frac{x^4}{4}+\frac{x^2}{2}+x\Big)\ln x - \int \frac{x^3}{4}\,dx - \int \frac{x}{2}\,dx - \int dx = \Big(\frac{x^4}{4}+\frac{x^2}{2}+x\Big)\ln x - \frac{x^4}{16} - \frac{x^2}{4} - x + c;$$

10. $$\int \sqrt[3]{x}\ln x\,dx = \begin{pmatrix} u=\ln x & dv=\sqrt[3]{x}\,dx \\ du=\frac{1}{x}\,dx & v=\int x^{\frac{1}{3}}\,dx=\frac{3}{4}x^{\frac{4}{3}} \end{pmatrix} = \frac{3}{4}x^{\frac{4}{3}}\ln x - \frac{3}{4}\int x^{\frac{4}{3}}\cdot\frac{1}{x}\,dx$$

$$= \frac{3}{4}\sqrt[3]{x^4}\ln x - \frac{3}{4}\int x^{\frac{1}{3}}dx = \frac{3}{4}\sqrt[3]{x^4}\ln x - \frac{9}{16}\sqrt[3]{x^4} + c = \frac{3}{4}\sqrt[3]{x^4}(\ln x - \frac{3}{4}) + c;$$

11. $$\int x\text{arctg }x\,dx = \begin{pmatrix} u=\text{arctg }x & dv=x\,dx \\ du=\frac{1}{1+x^2}\,dx & v=\int x\,dx=\frac{x^2}{2} \end{pmatrix} = \frac{x^2}{2}\text{arctg }x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\text{arctg }x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx = \frac{x^2}{2}\text{arctg }x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\text{arctg }x - \frac{1}{2}x + \frac{1}{2}\text{arctg }x + c;$$

12. $$\int \text{tg}^{\,2}x\frac{e^{\text{tg}x}}{\cos^2 x}\,dx = \begin{pmatrix} \text{tg}x=t \\ \frac{1}{\cos^2 x}\,dx=dt \end{pmatrix} = \int t^2e^t\,dt = \begin{pmatrix} u=t^2 & dv=e^t\,dt \\ du=2t\,dx & v=\int e^t\,dt=e^t \end{pmatrix}$$

$$= t^2e^t - 2\int te^t\,dt = \begin{pmatrix} u_1=t & dv_1=e^t dt \\ du_1=dx & v_1=\int e^t\,dt=e^t \end{pmatrix}$$

$$= t^2e^t - 2te^t + 2\int e^t\,dt = t^2e^t - 2te^t + 2e^t + c = \text{tg}^{\,2}xe^{\text{tg }x} - 2\text{tg }xe^{\text{tg }x} + 2e^{\text{tg }x} + c$$