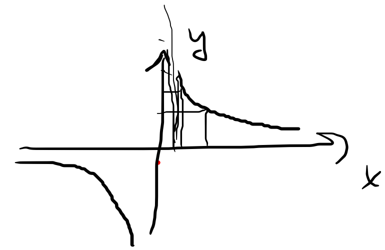
наставак граничних вредности

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}$$



1) $$\lim_{x\to\infty}\frac{3x^2+2x+4}{5x^5+3x^3+1} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\cancel{x^5}\left(\frac{3}{x^3}+\frac{2}{x^4}+\frac{4}{x^5}\right)}{\cancel{x^5}\left(5+\frac{3}{x^2}+\frac{1}{x^5}\right)}=\frac{0}{5}=0$$

2) $$\lim_{x\to\infty}\frac{5x^4+3x^2+1}{7x^3+8x^4} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\cancel{x^4}\left(5+\frac{3}{x^2}+\frac{1}{x^4}\right)}{\cancel{x^4}\left(\frac{7}{x}+8\right)}=\frac{5}{8}$$

3) $$\lim_{x\to\infty}\frac{13x^2+7x^4}{2x^3+4x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\cancel{x^4}\left(\frac{13}{x^2}+7\right)}{\cancel{x^4}\left(\frac{2}{x}+\frac{4}{x^3}\right)}= \text{"}\frac{7}{0^+}\text{"}=\infty$$

4) $$\lim_{x\to 0}\frac{x^3+x^5}{x^2+3x^4+x^5} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\cancel{x^2}(x+x^3)}{\cancel{x^2}(1+3x^2+x^3)}=\frac{0}{1}=0$$

5) $$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4} \overset{\frac{0}{0}}{=} \lim_{x\to 2}\frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)}=\frac{-1}{4}$$

$6-2$

$$x^2-5x+6=0$$

$$x_{1,2}=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm 1}{2}=\begin{cases}3\\2\end{cases}$$

6)

$$\lim_{x\to\infty}\left(\sqrt{x+3}-\sqrt{x}\right) \overset{\text{"}\infty-\infty\text{"}}{=} \lim_{x\to\infty}\left(\sqrt{x+3}-\sqrt{x}\right)\cdot\frac{\sqrt{x+3}+\sqrt{x}}{\sqrt{x+3}+\sqrt{x}} =$$

$$= \lim_{x\to\infty}\frac{\cancel{x}+3-\cancel{x}}{\sqrt{x+3}+\sqrt{x}} = \text{"}\frac{3}{\infty}\text{"} = 0$$

7)

$$\lim_{x\to\infty} x\left(x-\sqrt{x^2-1}\right) \overset{\text{"}\infty\cdot(\infty-\infty)\text{"}}{=} \lim_{x\to\infty} x\cdot\left(x-\sqrt{x^2-1}\right)\cdot\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} x\,\frac{x^2-(x^2-1)}{x+\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x}{x+\sqrt{x^2+1}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=}$$

$$= \lim_{x\to\infty}\frac{\cancel{x}}{\cancel{x}\left(1+\frac{\sqrt{x^2+1}}{x}\right)} = \lim_{x\to\infty}\frac{1}{1+\sqrt{\frac{x^2+1}{x^2}}} =$$

$$= \lim_{x\to\infty}\frac{1}{1+\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}} = \frac{1}{1+1} = \frac{1}{2}$$

8)

$$\lim_{x\to\infty} x\left(\sqrt{x^2-1}-\sqrt{x^2+1}\right) \overset{\infty\cdot(\infty-\infty)}{=} \lim_{x\to\infty} x\left(\sqrt{x^2-1}-\sqrt{x^2+1}\right)\cdot\frac{\sqrt{x^2+1}+\sqrt{x^2+1}}{\sqrt{x^2-1}+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} x\,\frac{x^2-1-(x^2+1)}{\sqrt{x^2-1}+\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{-2x}{\sqrt{x^2-1}+\sqrt{x^2+1}} \overset{\frac{-\infty}{\infty}}{=}$$

$$= \lim_{x\to\infty}\frac{-2x}{\sqrt{x^2\left(1-\frac{1}{x^2}\right)}+\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \lim_{x\to\infty}\frac{-2x}{x\sqrt{1-\frac{1}{x^2}}+x\sqrt{1+\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{-2\cancel{x}}{\cancel{x}\left(\sqrt{1-\cancel{\frac{1}{x^2}}^{\,0}}+\sqrt{1+\cancel{\frac{1}{x^2}}^{\,0}}\right)} = \frac{-2}{1+1} = -1$$

9) $\lim\limits_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} \overset{\frac{0}{0}}{=} \lim\limits_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} \cdot \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} =$

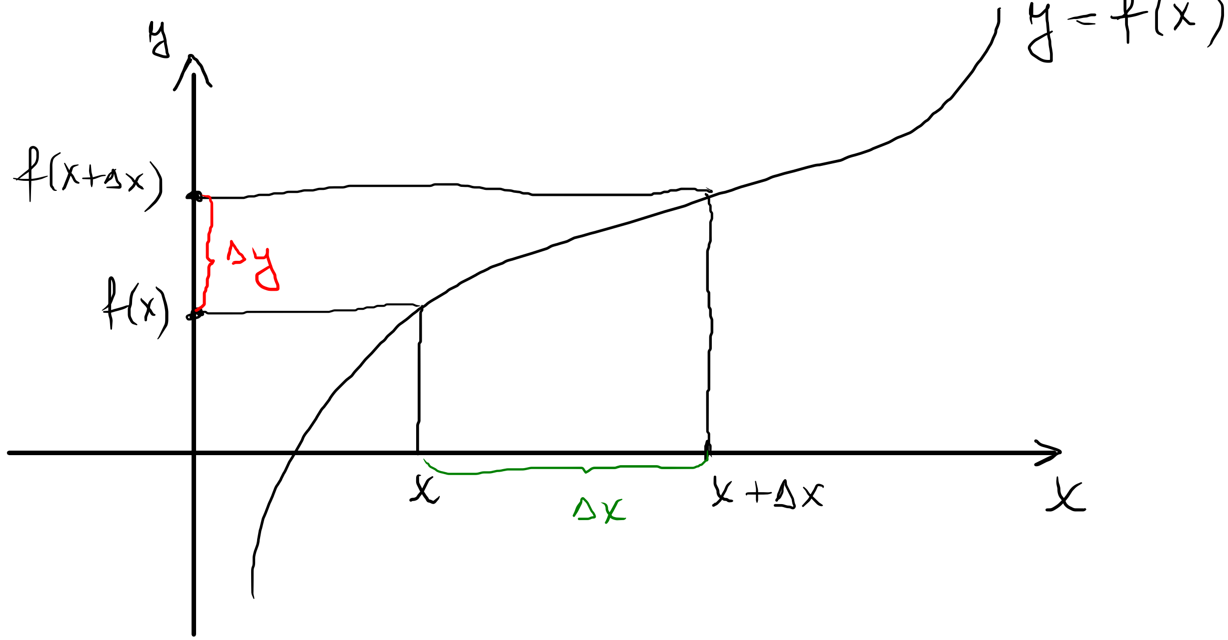
$= \lim\limits_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} = \lim\limits_{x\to -1} \frac{\cancel{x+1}}{\cancel{(x+1)}(\sqrt{x+5}+2)} = \frac{1}{2+2} = \frac{1}{4}$

10) $\lim\limits_{x\to 1} \frac{\sqrt{3x+1}-\sqrt{4x}}{1-\sqrt{x}} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 1} \frac{\sqrt{3x+1}-\sqrt{4x}}{1-\sqrt{x}} \cdot \frac{\sqrt{3x+1}+\sqrt{4x}}{\sqrt{3x+1}+\sqrt{4x}} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$

$= \lim\limits_{x\to 1} \frac{(3x+1-4x)(1+\sqrt{x})}{(1-x)(\sqrt{3x+1}+\sqrt{4x})} = \lim\limits_{x\to 1} \frac{\cancel{(1-x)}(1+\sqrt{x})}{\cancel{(1-x)}(\sqrt{3x+1}+\sqrt{4x})}$

$= \frac{1+1}{2+2} = \frac{1}{2}$

Изводи



y=f(x) неко је дефинисано на интервалу $(a,b) \in \mathbb{R}$

$\Delta x \neq 0$ прираштај аргумента у тачки $x \in (a,b)$

Δy – прираштај функције

$$\Delta y = f(x+\Delta x) - f(x)$$

$$f'(x) = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

1. $y = x^2$ $\quad y' = ?$ $\quad y = f(x)$

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2\cdot x\Delta x + \Delta x^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

Правила за наоѓање изводи:

- Нека $f(x)$ и $g(x)$ имају извод у точки x; Тогаш е:

1. $(a f(x))' = a f'(x)$, $a = const$ $\qquad y = 3x^2 \quad y' = (3x^2)' = 3(x^2) = 3\cdot 2x$
2. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
3. $(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ $\qquad g(x) \neq 0$

$$(f(x)g(x)h(x))' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

ТАБЛИЦА ИЗВОДА ЕЛЕМЕНТАРНИХ ФУНКЦИЈА (НА ОБЛАСТИ ДЕФИНИСАНОСТИ)

1. $c' = 0$, $c = const$ $\qquad 2' = 0$, $8' = 0$, $(-7)' = 0$

2. $\boxed{(x^{\alpha})' = \alpha x^{\alpha-1}}$

$$(x^7)' = 7x^6 \qquad \left(\frac{1}{x^7}\right)' = (x^{-7})' = -7x^{-8}$$

$$(x)' = 1 \qquad \left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2}$$

$$(x^{105})' = 105x^{104}$$

$$\left(\frac{1}{\sqrt{x}}\right)' = \left(x^{-\frac{1}{2}}\right)' = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$\left(\frac{1}{\sqrt[5]{x^{11}}}\right)' = \left(x^{-\frac{11}{5}}\right)' = -\frac{11}{5}x^{-\frac{11}{5}-1} = -\frac{11}{5}x^{-\frac{16}{5}}$$

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\left(\sqrt[3]{x^5}\right)' = \left(x^{\frac{5}{3}}\right)' = \frac{5}{3}x^{\frac{5}{3}-1} = \frac{5}{3}x^{\frac{2}{3}} = \frac{5}{3}\sqrt[3]{x^2}$$

3) $(\log_a x)' = \dfrac{1}{x \ln a}$ $\qquad (\log_2 x)' = \dfrac{1}{x \ln 2}$

4) $(\ln x)' = \dfrac{1}{x}$

5) $(a^x)' = a^x \ln a$ $\qquad (2^x)' = 2^x \ln 2$

6) $(e^x)' = e^x$

7) $(\sin x)' = \cos x$

8) $(\cos x)' = -\sin x$

9) $(\operatorname{tg} x)' = \dfrac{1}{\cos^2 x}$

10) $(\operatorname{ctg} x)' = -\dfrac{1}{\sin^2 x}$

11) $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

12) $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

13) $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$

14) $(\operatorname{arcctg} x)' = -\dfrac{1}{1+x^2}$

1)

$$y = 5x^2 + 2x^3 + 3x$$

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

$$y' = (5x^2 + 2x^3 + 3x)'$$

$$= (5x^2)' + (2x^3)' + (3x)'$$

$$= 5(x^2)' + 2(x^3)' + 3(x)'$$

$$= 5 \cdot 2x + 2 \cdot 3x^2 + 3 \cdot 1$$

$$= 10x + 6x^2 + 3$$

$$y' = 10x + 6x + 3$$

2)

$$y = 2\sqrt{x} - \frac{2}{x} + \sqrt[3]{4}$$

$$y' = 2(\sqrt{x})' - 2\left(\frac{1}{x}\right)' + (\sqrt[3]{4})' \quad = \text{const}$$

$$= 2\left(x^{\frac{1}{2}}\right)' - 2\left(x^{-1}\right)' + 0$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} x^{\frac{1}{2}-1} - 2 \cdot (-1) x^{-1-1}$$

$$= x^{-\frac{1}{2}} + 2 \cdot x^{-2}$$

$$= \frac{1}{\sqrt{x}} + \frac{2}{x^2}$$

3) $y = x^3 + \sin x$

$y' = 3x^2 + \cos x$

4) $y = \sin x + \cos x$

$y' = \cos x - \sin x$

5) $y = x^3 \cdot \sin x$

$y' = (x^3)' \sin x + x^3 \cdot (\sin x)'$

$= 3x^2 \sin x + x^3 \cos x$

6) $y = e^x \sin x$

$y' = (e^x)' \sin x + e^x (\sin x)'$

$= e^x \sin x + e^x \cos x$

7,

$$y = (x^3 + x^2 + x + 1) \cdot e^x$$

$$y' = (x^3+x^2+x+1)' e^x + (x^3+x^2+x+1)(e^x)'$$

$$= (3x^2+2x+1)e^x + (x^3+x^2+x+1)e^x$$

$$= (x^3+4x^2+3x+2)e^x$$

8,

$$y = \underbrace{x^2 \cos x} + \underbrace{\sin x}$$

$$y' = (x^2 \cdot \cos x)' + (\sin x)'$$

$$= (x^2)' \cos x + x^2 \cdot (\cos x)' + \cos x$$

$$= 2x \cos x + x^2(-\sin x) + \cos x$$

9) $y = \dfrac{\sin x}{x^2}$

$$y' = \frac{(\sin x)' x^2 - \sin x (x^2)'}{(x^2)^2}$$

$$= \frac{\cos x \cdot x^2 - \sin x \cdot 2x}{x^4}$$

$$= \frac{\cancel{x}(x\cos x - 2\sin x)}{x^{\cancel{4}\,3}}$$

$$= \frac{x\cos x - 2\sin x}{x^3}$$

$\cos x \cdot x^2 \neq \cos x^3$

10)

$$y = \frac{x}{x^2+2}$$

$$y' = \frac{x' \cdot (x^2+2) - x\,(x^2+2)'}{(x^2+2)^2}$$

$$= \frac{x^2+2 - x \cdot 2x}{(x^2+2)^2}$$

$$= \frac{2-x^2}{(x^2+2)^2}$$

11)

$$y = \frac{\ln x}{x}$$

$$y' = \frac{(\ln x)' \cdot x - \ln x \cdot (x)'}{x^2}$$

$$y' = \frac{\cancel{\frac{1}{x} \cdot x} - \ln x \cdot 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

12)

$$y = \frac{1-\ln x}{1+\ln x}$$

$$y' = \frac{(1-\ln x)'(1+\ln x) - (1-\ln x)\cdot(1+\ln x)'}{(1+\ln x)^2}$$

$$= \frac{-\frac{1}{x}(1+\ln x) - (1-\ln x)\frac{1}{x}}{(1+\ln x)^2}$$

13)

$$y = \frac{x\sin x}{1+\operatorname{tg} x}$$

$$y' = \frac{(x\cdot\sin x)'(1+\operatorname{tg} x) - x\sin x(1+\operatorname{tg} x)'}{(1+\operatorname{tg} x)^2}$$

$$= \frac{(x'\sin x + x\cdot(\sin x)')(1+\operatorname{tg} x) - x\sin x\frac{1}{\cos^2 x}}{(1+\operatorname{tg} x)^2}$$

$$= \frac{(\sin x + x\cos x)(1+\operatorname{tg} x) - x\cdot\sin x\frac{1}{\cos^2 x}}{(1+\operatorname{tg} x)^2}$$

14)

$$y = \operatorname{tg} x = \frac{\sin x}{\cos x}$$

$$y' = (\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{(\sin x)' \cdot \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$g(x)$ што извод у тачки x

$f(x)$ што извод у тачки $u = g(x)$

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

1) $y = \sin(2x+1)$

$y' = \cos(2x+1) \cdot (2x+1)'$

$= \cos(2x+1) \cdot 2$

$y = \sin x$

$y' = \cos x$

2) $y = e^{3x}$

$y' = e^{3x} \cdot (3x)'$

$= e^{3x} \cdot 3$

3)

$$y = \sin^2 x = (\sin x)^2 = \sin x \cdot \sin x$$

$$y' = 2 \cdot \sin x \cdot (\sin x)'$$

$$= 2 \cdot \sin x \cos x$$

4)

$$y = \sin x^2 = \sin(x \cdot x)$$

$$y' = \cos x^2 \cdot (x^2)'$$

$$= \cos x^2 \cdot 2x$$

5)

$$y = \operatorname{arctg} x^2$$

$$y' = \frac{1}{1+(x^2)^2} \cdot (x^2)'$$

$$' = \frac{1}{1+x^4} \cdot 2x$$

6)

$$y = \ln(\sin x)$$

$$y' = \frac{1}{\sin x} \cdot (\sin x)'$$

$$y' = \frac{1}{\sin x} \cdot \cos x$$

7)

$$y = \ln \sqrt{\frac{1+3x}{1-3x}}$$

$$y' = \frac{1}{\sqrt{\frac{1+3x}{1-3x}}} \left(\sqrt{\frac{1+3x}{1-3x}}\right)'$$

$$= \sqrt{\frac{1-3x}{1+3x}} \left(\left(\frac{1+3x}{1-3x}\right)^{\frac{1}{2}}\right)'$$

$$=$$

$$= \sqrt{\frac{1-3x}{1+3x}} \cdot \frac{1}{2} \left(\frac{1+3x}{1-3x}\right)^{-\frac{1}{2}} \left(\frac{1+3x}{1-3x}\right)'$$

$$= \sqrt{\frac{1-3x}{1+3x}} \cdot \frac{1}{2} \sqrt{\frac{1-3x}{1+3x}} \; \frac{(1+3x)'(1-3x) - (1+3x)(1-3x)'}{(1-3x)^2}$$

$$= \frac{1-3x}{1+3x} \cdot \frac{1}{2} \; \frac{3(1-3x) - (1+3x)(-3)}{(1-3x)^2}$$

8)

$$y = \sin^2(\ln x) = (\sin(\ln x))^2$$

$$y' = 2\cdot(\sin(\ln x))\cdot(\sin(\ln x))'$$

$$= 2\sin(\ln x)\cdot\cos(\ln x)\cdot(\ln x)'$$

$$= 2\sin(\ln x)\cdot\cos(\ln x)\cdot\frac{1}{x}$$

9)

$$y = \sin e^{x+3}$$

$$y' = \cos e^{x+3}\cdot(e^{x+3})'$$

$$= \cos e^{x+3}\cdot e^{x+3}\cdot(x+3)'$$

$$= \cos e^{x+3}\cdot e^{x+3}$$

10)

$$y = \ln\left(\sin e^{2x}\right)$$

$$y' = \frac{1}{\sin e^{2x}} \cdot \left(\sin e^{2x}\right)'$$

$$= \frac{1}{\sin e^{2x}} \cdot \cos e^{2x} \cdot \left(e^{2x}\right)'$$

$$= \operatorname{ctg} e^{2x} \cdot e^{2x} \cdot (2x)'$$

$$= \operatorname{ctg} e^{2x} \cdot e^{2x} \cdot 2$$

11,

$$y = \ln^3(\operatorname{arcctg}(x+3)) = (\ln(\operatorname{arcctg}(x+3)))^3$$

$$y' = 3(\ln(\operatorname{arcctg}(x+3)))^2 \cdot (\ln(\operatorname{arcctg}(x+3)))'$$

$$= 3(\ln(\operatorname{arcctg}(x+3)))^2 \cdot \frac{1}{\operatorname{arcctg}(x+3)} \cdot (\operatorname{arcctg}(x+3))'$$

$$= \frac{3(\ln(\operatorname{arcctg}(x+3)))^2}{\operatorname{arcctg}(x+3)} \cdot \frac{1}{1+(x+3)^2} \cdot (x+3)'$$

$$= \frac{3(\ln(\operatorname{arcctg}(x+3)))^2}{\operatorname{arcctg}(x+3) \cdot (1+(x+3)^2)}$$

12)

$$y = \ln\left(x + \sqrt{1+x^2}\right)$$

$$\boxed{\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}}$$

$$\left(\sqrt{x}\right)' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$y' = \frac{1}{x+\sqrt{1+x^2}}\left(x+\sqrt{1+x^2}\right)'$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}\cdot\left(1+x^2\right)'\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x\right)$$

$$= \frac{1}{\cancel{x+\sqrt{1+x^2}}}\;\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$y'' = (y')'$$

$$= \left(\frac{1}{\sqrt{1+x^2}}\right)'$$

$$= \left(\left(1+x^2\right)^{-\frac{1}{2}}\right)'$$

$$= -\frac{1}{2}\left(1+x^2\right)^{-\frac{3}{2}}\cdot\left(1+x^2\right)'$$

$$= -\frac{1}{\cancel{2}\sqrt{(1+x^2)^3}}\cdot\cancel{2}x$$

13, $y=(x-2)\cdot e^{2x}$

$$y'=(x-2)'e^{2x}+(x-2)\left(e^{2x}\right)'$$
$$=e^{2x}+(x-2)\,e^{2x}\cdot(2x)'$$
$$=e^{2x}+(x-2)e^{2x}\cdot 2$$
$$=(1+2x-4)e^{2x}$$
$$=(2x-3)\,e^{2x}$$

$$y''=(y')'=\left((2x-3)e^{2x}\right)'$$
$$=(2x-3)'\,e^{2x}+(2x-3)\left(e^{2x}\right)'$$
$$=2e^{2x}+(2x-3)\,e^{2x}\cdot(2x)'$$
$$=2e^{2x}+(2x-3)\,e^{2x}\cdot 2$$

14)

$$y = x^2 \cdot \ln(\sin x)$$

$$y' = (x^2)' \ln(\sin x) + x^2 (\ln(\sin x))'$$

$$= 2x \ln(\sin x) + x^2 \frac{1}{\sin x} \cdot (\sin x)'$$

$$= 2x \ln(\sin x) + x^2 \frac{1}{\sin x} \cos x$$

$$= \underbrace{2x \cdot \ln(\sin x)} + \underbrace{x^2 \operatorname{ctg} x}$$

$$y'' = (y')' = \underbrace{(2x)' \ln(\sin x) + 2x \cdot (\ln(\sin x))'} + \underbrace{(x^2)' \operatorname{ctg} x + x^2 (\operatorname{ctg} x)'}$$

$$= 2 \ln(\sin x) + 2x \cdot \frac{1}{\sin x} \cdot (\sin x)' + 2x \operatorname{ctg} x + x^2 \left(-\frac{1}{\sin^2 x}\right)$$

$$= 2 \ln(\sin x) + 2x \frac{1}{\sin x} \cos x + 2x \operatorname{ctg} x - x^2 \frac{1}{\sin^2 x}$$

Неодређени интеграли

$f(x)$ дефинисана на интервалу I

$F(x)$ примитивна функција функције $f(x)$ на I ако је $F(x)$ таква да што извод и да важи

$$F'(x) = f(x), \quad \forall x \in I$$

$$f(x) = \cos x \longrightarrow F_1(x) = \sin x$$

$$F_2(x) = \sin x + 3$$

$$F_2(x) = \sin x - 5$$

$$F(x) = \sin x + c$$

$$F'(x) = (\sin x + c)' = \cos x = f(x)$$

Скуп свих примитивних функција функције f на неком интервалу I назива се НЕОДРЕЂЕНИ ИНТЕГРАЛ функције f на том интервалу и означава се

$$\int f(x)\,dx$$

напомена:

$$y' = \frac{dy}{dx}$$

$$\boxed{\int f(x)\,dx = F(x) + C}$$

$C = const$

Особине неодређеног интеграла:

1) $\left(\int f(x)\,dx\right)' = f(x)$

$\left(\int f(x)\,dx\right)' = \left(F(x)+c\right)' = F'(x) = f(x)$

2) $\int f'(x)\,dx = f(x) + c$

3) $\int \alpha f(x)\,dx = \alpha \int f(x)\,dx, \quad \alpha = const$

4) $\int \left(f(x) \pm g(x)\right) dx = \int f(x)\,dx \pm \int g(x)\,dx$

Таблица интегралов:

$\int x^2 dx =$

$\left(\frac{x^3}{3}\right)' = x^2$

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

2) $\int \frac{dx}{x} = \ln|x| + C$, $x \neq 0$ $\quad (\ln x)' = \frac{1}{x}$

$(x^\alpha)' = \alpha x^{\alpha - 1}$

3) $\int \sin x \, dx = -\cos x + C$ $\quad (-\cos x)' = -(-\sin x) = \sin x$

$(x^3)' = 3x^2$

4) $\int \cos x \, dx = \sin x + C$ $\quad (\sin x)' = \cos x$

5) $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$ $\quad (\operatorname{tg} x)' = \frac{1}{\cos^2 x}$

6) $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$ $\quad (\operatorname{ctg} x) = \frac{-1}{\sin^2 x}$

7) $\int e^x dx = e^x + C$

8 $\int a^x dx = \frac{a^x}{\ln x} + C$

9 $\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$ $\quad (\operatorname{arctg} x)' = \frac{1}{1+x^2} = (-\operatorname{arcctg} x)'$

10 $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$ $\quad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} = (-\arccos x)'$

11 $\boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C}$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int x\, dx = \frac{x^2}{2} + C$$

$$\int dx = x + C$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$\int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-6}}{-6} + C$$

$$\boxed{\int \frac{1}{x} dx = \ln|x| + C}$$

$$\int \sqrt{x}\, dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$\int \sqrt[5]{x^7}\, dx = \int x^{\frac{7}{5}} dx = \frac{x^{\frac{12}{5}}}{\frac{12}{5}} + C$$

$$\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$\int \frac{1}{\sqrt[11]{x^{13}}} dx = \int x^{-\frac{13}{11}} dx = \frac{x^{-\frac{2}{11}}}{-\frac{2}{11}} + C$$

1) $\int \left(6x^2 + 8x + 3 + \sqrt{x} - \frac{1}{x^3} + \frac{1}{4\sqrt{x}}\right) dx$

$= 6\int x^2 dx + 8\int x\,dx + 3\int dx + \int \sqrt{x}\,dx - \int \frac{1}{x^3} dx + \int \frac{1}{4\sqrt{x}} dx$

$= 6\frac{x^3}{3} + 8\frac{x^2}{2} + 3x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{-2}}{-2} + \frac{x^{\frac{3}{4}}}{\frac{3}{4}} + C$

2) $\int \left(x^3 + 2\sin x - \frac{1}{\sin^2 x} + \frac{1}{1+x^2}\right) dx$

$= \frac{x^4}{4} + 2(-\cos x) - (-\operatorname{ctg} x) + \operatorname{arctg} x + C$

3) $\int \left(e^x + \frac{1}{x} + \frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{1-x^2}}\right) dx$

$= e^x + \ln|x| + \ln\left|x + \sqrt{x^2+1}\right| - \arcsin x + C$

4) $\int \frac{x^2 \overset{+1-1}{}}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx = \int \left(1 - \frac{1}{x^2+1}\right)dx$

$= \int dx - \int \frac{dx}{x^2+1} = x - \operatorname{arctg} x + C$

5) $\int tg^2 x\, dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx$

$= \int \left(\frac{1}{\cos^2 x} - 1\right)dx = \int \frac{1}{\cos^2 x}dx - \int dx$

$= tg\, x - x + C$

Смена переменных

$\varphi(t)$

$$\boxed{\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt}$$

$x = \varphi(t) \quad |^{d}$

$dx = \varphi'(t)\,dt$

$y' = \dfrac{dy}{dx}$

$dy = y'dx$

1)

$$\int (1+x)^6 \, dx = \int t^6 \, dt = \frac{t^7}{7} + C$$

смена: $1+x = t \quad |d$

$dx = dt$

$$= \frac{1}{7}(1+x)^7 + C$$

$$\int t^6 \, dt = \frac{t^7}{7} + C$$

2,

$$\int (2x+3)^{15} \, dx = \int t^{15} \cdot \frac{1}{2} \, dt = \frac{1}{2} \int t^{15} \, dt = \frac{1}{2} \frac{t^{16}}{16} + C$$

$2x+3 = t$

$2dx = dt$

$dx = \frac{1}{2} dt$

$$= \frac{1}{32} (2x+3)^{16} + C$$

$$\left(\frac{1}{32}(2x+3)^{16} + C\right)' = \frac{1}{\cancel{32}} \cdot \cancel{16}(2x+3)^{15} \cdot \cancel{2} = (2x+3)^{15}$$

3)
$$\int \frac{dx}{1-2x} = \int \frac{1}{t} \cdot \left(-\frac{1}{2}\right) dt = -\frac{1}{2} \int \frac{1}{t} dt$$

$$= -\frac{1}{2} \ln|t| + C = -\frac{1}{2} \ln|1-2x| + C$$

$1-2x = t$

$-2dx = dt$

$dx = -\frac{1}{2} dt$

4)
$$\int \frac{dx}{\sqrt{5x-2}} = \int \frac{1}{\sqrt{t}} \cdot \frac{1}{5} dt = \frac{1}{5} \int t^{-\frac{1}{2}} dt = \frac{1}{5} \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{5} \sqrt{5x-2} + C$$

$5x-2 = t$

$5dx = dt$

$dx = \frac{1}{5} dt$

5) $\int e^{3x}dx = \int e^t \cdot \frac{1}{3}dt = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + C$

$3x = t$

$3dx = dt$

$dx = \frac{1}{3}dt$

$= \frac{1}{3}e^{3x} + C$

6) $\int \sin 5x\,dx = \int \sin t \cdot \frac{1}{5}dt = \frac{1}{5}\int \sin t\,dt$

$= \frac{1}{5}(-\cos t) + C$

$= -\frac{1}{5}\cos 5x + C$

$5x = t$

$5dx = dt$

$dx = \frac{1}{5}dt$

7, $\int tg\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C$

$\cos x = t$

$-\sin x\,dx = dt$

$\sin x\,dx = -dt$

$dx = \frac{dt}{-\sin x}$

напомена : $\sin x = u$, $\cancel{\cos x\,dx} = du$ $\cancel{\int u \frac{dx}{\cos x}}$

$\int \frac{\cancel{\sin x}}{t} \cdot \frac{dt}{\cancel{-\sin x}} = \int -\frac{dt}{t}$

$\int \frac{dx}{\sqrt{x^2+c}} = \ln|x+\sqrt{x^2+1}| + C$

8, $\int \frac{x\,dx}{\sqrt{x^4+1}} = \int \frac{\frac{1}{2}\,dt}{\sqrt{t^2+1}} = \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right| + C$

$x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{1}{2}\,dt$

$= \frac{1}{2}\ln\left|x^2+\sqrt{x^4+1}\right| + C$

9) $\int \frac{dx}{9+x^2} = \frac{1}{9}\int \frac{dx}{1+\frac{x^2}{9}}$ $\qquad \int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$

$= \frac{1}{9}\int \frac{dx}{1+\left(\frac{x}{3}\right)^2} = \frac{1}{\cancel{9}_3}\int \frac{\cancel{3}\,dt}{1+t^2} = \frac{1}{3}\operatorname{arctg} t + C$

$\frac{x}{3} = t$ $\qquad = \frac{1}{3}\operatorname{arctg}\frac{x}{3} + C$

$x = 3t$

$dx = 3\,dt$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

10)

$$\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{16\left(1-\frac{x^2}{16}\right)}} =$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \arcsin x + C$$

$$= \frac{1}{4}\int \frac{dx}{\sqrt{1-\left(\frac{x}{4}\right)^2}} = \frac{1}{\cancel{4}}\int \frac{\cancel{4}dt}{\sqrt{1-t^2}} = \arcsin t + C$$

$$= \arcsin \frac{x}{4} + C$$

$\frac{x}{4} = t$

$x = 4t$

$dx = 4dt$

Парцијална интеграција

$u = u(x)$

$v = v(x)$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$u \longrightarrow du \qquad\qquad dv \longrightarrow v$

1) $\int \ln x\,dx = \ln x \cdot x - \int \cancel{x}\,\frac{1}{\cancel{x}}\,dx = x\ln x - x + C$

$u = \ln x \qquad dv = dx$

$\boxed{du = \frac{1}{x}dx} \qquad \boxed{v = x}$

$\int dx = x + C$

2) $\int \underline{x}\,\underline{e^x}\,dx = xe^x - \int e^x dx = xe^x - e^x + C$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

3) $\int \underbrace{x\cos x\,dx}_{u\,dv} = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\left.\begin{aligned} &\int P_n(x)\,e^{\alpha x}\,dx \\ &\int P_n(x)\,\sin\alpha x\,dx \\ &\int P_n(x)\,\cos\alpha x\,dx \end{aligned}\right\} \quad u = P_n(x) \qquad dv = \begin{cases} e^{\alpha x}\,dx \\ \sin\alpha x\,dx \\ \cos\alpha x\,dx \end{cases}$$

4) $\int (x+1)^2 \cos x \, dx = (x+1)^2 \sin x - \int \sin x \cdot 2(x+1) \, dx$

$u = (x+1)^2 \qquad dv = \cos x \, dx$

$du = 2(x+1)\,dx \qquad v = \sin x$

$= (x+1)^2 \sin x - 2 \int (x+1) \sin x \, dx$

$u_1 = x+1 \qquad dv_1 = \sin x \, dx$

$du_1 = dx \qquad v_1 = -\cos x$

$= (x+1)^2 \sin x - 2\left((x+1)(-\cos x) - \int (-\cos x) \, dx \right)$

$= (x+1)^2 \sin x + 2(x+1)\cos x - 2\int \cos x \, dx$

$= (x+1)^2 \sin x + 2(x+1)\cos x - 2\sin x + C$

5, $\int x^2 \sin 5x\, dx = x^2\left(-\frac{1}{5}\cos 5x\right) - \int\left(-\frac{1}{5}\cos 5x \cdot 2x\right)dx$

$u = x^2$ $\quad$ $dv = \sin 5x\, dx$ $\quad$ $\int \sin \underline{5x}\, dx = \frac{1}{5}\int \sin t\, dt$

$du = 2x\, dx$ $\quad$ $v = -\frac{1}{5}\cos 5x$

$5x = t$ $\quad$ $= -\frac{1}{5}\cos t + C$

$5dx = dt$

$dx = \frac{1}{5}dt$ $\quad$ $= -\frac{1}{5}\cos 5x + C$

$= -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\int \underline{x}\, \underline{\cos 5x}\, dx$

$u_1 = x$ $\quad$ $dv_1 = \cos 5x\, dx$

$du_1 = dx$ $\quad$ $v_1 = \frac{1}{5}\sin 5x$

$\int \cos \alpha x\, dx = \frac{1}{\alpha}\sin \alpha x + C$

$\int \sin \alpha x\, dx = -\frac{1}{\alpha}\cos \alpha x + C$

$= -\frac{1}{5}x^2\cos 5x + \frac{2}{5}\left(x \cdot \frac{1}{5}\sin 5x - \int \frac{1}{5}\sin 5x \cdot dx\right)$

$= -\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x - \frac{2}{25}\left(-\frac{1}{5}\cos 5x\right) + C$

6) $\int x \operatorname{arctg} x \, dx = \frac{x^2}{2} \operatorname{arctg} x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx$

$u = \operatorname{arctg} x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^2}{2}$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^{2+1-1}}{1+x^2} dx$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} dx$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2}{2} \operatorname{arctg} x - \int \frac{1}{2} dx + \frac{1}{2} \int \frac{1}{1+x^2} dx$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x + C$

$$r(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

$p(x), q(x)$ – полиноми

права

$\deg(p) < \deg(q)$

$$\frac{1}{x+3}$$

$$\frac{2x}{x^2+5}$$

$$\frac{x^4+3x}{x^5+7x^7}$$

неправа

$\deg(p) \geq \deg(q)$

$$\frac{x^2+3}{x+1}$$

$$\frac{x^7+3x^2+1}{x^2+2}$$

$$\frac{x^2+4}{x^2+7}$$

$$\frac{x^2+3}{x+1} = (x-1) + \frac{4}{x+1}$$

$$(x^2+3) : (x+1) = \underline{\underline{x-1}}$$
$$-(x^2+x)$$
$$-x+3$$
$$-(-x-1)$$
$$\underline{\underline{4}}$$

$$\frac{A}{(x-a)^k}, \qquad \frac{Bx+C}{(\underbrace{x^2+px+q}_{\text{няма реални нули}})^n}$$

$$\frac{1}{(x-a)^{k}(x^2+px+q)^{n}} = \underbrace{\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_k}{(x-a)^k}}_{k}$$

$$+ \underbrace{\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \dots + \frac{B_nx+C_n}{(x^2+px+q)^n}}_{n}$$

1) $$\frac{1}{(x-1)^{2}(x^2+5)^{3}} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+5} + \frac{Ex+F}{(x^2+5)^2} + \frac{Mx+N}{(x^2+5)^3}$$

$$\frac{1}{\underline{\underline{x^2}}\,\underline{\underline{(x^2+1)}}} = \boxed{\frac{A}{x} + \frac{B}{x^2}} + \frac{Cx+D}{x^2+1}$$

$$x^2 \to (x-0)^2$$

$$\frac{Ax+B}{x^2}$$

има реални нули

2) $$\int \frac{1}{x^2-4}\,dx = \int \frac{1}{(x-2)(x+2)}\,dx = \int \left(\frac{\frac{1}{4}}{x-2} + \frac{-\frac{1}{4}}{x+2} \right) dx = \frac{1}{4}\int \frac{1}{x-2}\,dx - \frac{1}{4}\int \frac{1}{x+2}\,dx$$

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \quad / \cdot (x-2)(x+2)$$

$$1 = A(x+2) + B(x-2)$$

$$1 = Ax + 2A + Bx - 2B$$

$$1 = (A+B)x + 2A - 2B$$

$$\begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \cdot(-2)$$

$$\begin{cases} A+B=0 \\ -4B=1 \end{cases}$$

$$B = -\frac{1}{4}$$

$$A = \frac{1}{4}$$

$$\frac{1}{(x-2)(x+2)} = \frac{\frac{1}{4}}{x-2} + \frac{-\frac{1}{4}}{x+2}$$

смена $\quad x-2=t,\ dx=dt \qquad x+2=u,\ dx=du$

$$= \frac{1}{4}\int \frac{1}{t}\,dt - \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|t| - \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

3) $$\int \frac{1}{x^2-3x+2}\,dx = \int \frac{1}{(x-2)(x-1)}\,dx = *$$

$$x_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} \begin{cases} 2 \\ 1 \end{cases}$$

$$\frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \quad / (x-2)(x-1)$$

$$1 = A(x-1) + B(x-2)$$

$$1 = Ax - A + Bx - 2B$$

$$1 = (A+B)x + (-A-2B)$$

$$\begin{aligned} A+B &= 0 \\ -A-2B &= 1 \end{aligned} \quad +$$

$$\begin{aligned} A+B &= 0 \\ -B &= 1 \end{aligned}$$

$B = -1 \qquad A = 1$

$$\frac{1}{(x-2)(x-1)} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$* \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx$$

$$= \int \frac{1}{x-2}\,dx - \int \frac{1}{x-1}\,dx$$

Smjena:

$x-2=t \qquad x-1=u$

$dx=dt \qquad dx=du$

$$= \int \frac{dt}{t} - \int \frac{du}{u}$$

$$= \ln|t| - \ln|u| + C$$

$$= \ln|x-2| - \ln|x-1| + C$$

4) $\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx = \int \frac{1}{t^2+\frac{3}{4}}dt =$

смена
$x+\frac{1}{2}=t$
$dx=dt$

$x_{1,2}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}\notin R$

$x^2+x+1 = x^2+2x\boxed{\frac{1}{2}}+(\frac{1}{2})^2-(\frac{1}{2})^2+1$

$(A+B)^2=A^2+2AB+B^2$

$=(x+\frac{1}{2})^2+\frac{3}{4}$

$=\frac{1}{\frac{3}{4}}\int\frac{1}{\frac{t^2}{\frac{4}{3}}+1}dt=\frac{4}{3}\int\frac{dt}{\left(\frac{t}{\frac{2}{\sqrt{3}}}\right)^2+1}=\frac{4}{3}\int\frac{\frac{2}{\sqrt{3}}du}{u^2+1}=$

$=\frac{8}{3\sqrt{3}}\operatorname{arctg} u+C$

$\frac{t}{\frac{2}{\sqrt{3}}}=u \Rightarrow t=\frac{2}{\sqrt{3}}u \Rightarrow dt=\frac{2}{\sqrt{3}}du$

$=\frac{8}{3\sqrt{3}}\operatorname{arctg}\frac{t}{\frac{2}{\sqrt{3}}}+C=\frac{8}{3\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{2}{\sqrt{3}}}+C$