Linearne transformacije

January 16, 2021

1. Za date funkcije ispitati da li su linearne transformacije i za one koje jesu naći matricu i rang.

   1.1 $f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x,y) = (x^2 - y, x+y)$ NIJE

$$\to\quad f:\mathbb{R}^n \to \mathbb{R}^m,\quad f(x_1,\dots,x_n) = (\underbrace{\_\_\_,\ \_\_\_,\ \dots\ \_\_\_}_{m})$$

$$t_1x_1 + t_2x_2 + \dots + t_nx_n$$

1.2 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x, y) = (2x - y, 3x, y + 1)$

NIJE

$3x + 0y$

$y + 1$

$t_1 x + t_2 y$, $t_1, t_2 \in \mathbb{R}$

1.3 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x, y, z) = (x - y + 2z, -x + 3y + z)$ JESTE

$$M_f = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & 1 \end{bmatrix}$$

1.4 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f(x, y, z) = \underline{3x + 2y - z}$ JESTE

$$M_f = \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

1.5 $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x,y,z) = (x+y, 0)$

0

$\cancel{y+1}$

JESTE

$0 \cdot x + 0 \cdot y$

$$M_f = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

1.6 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (x, \cos(xy))$

$1 \cdot x + 0 \cdot y$

$\cancel{\underline{\cos}(x \cdot y)}$

NIJE

1.7 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = \sqrt{x}$

$\cancel{\sqrt{x}}$

NIJE

$t_1 x + t_2 y \,,\ t_1, t_2 \in \mathbb{R}$

1.8 $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \underbrace{5x}$

$$M_f = \begin{bmatrix} 5 \end{bmatrix}$$

1.9 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = \left(\underbrace{x}_{w}, \underbrace{y}_{w}, \underbrace{\sqrt{x^2+y^2}}\right)$

$\cancel{\sqrt{x^2+y^2} \neq \sqrt{x^2}+\sqrt{y^2}}$

$\sqrt{x^2+y^2}$ – FUNKCJA

t_1x+t_2y NIE

1.10 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (\ln 2 \cdot x, x + y)$

JESTE

$\ln 2 \cdot x$

BROJ

BROJ $\cdot x$

$t_1 x + t_2 y$,

$$M_f = \begin{bmatrix} \ln 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$t_1 = \ln 2$

$t_2 = 0$

2. Za sledeće funkcije diskutovari po realnim parametrima kada su linearne transformacije i u slučaju kada jesu naći njihove matrice i odrediti rang.

   2.1 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x, y, z) = (\underbrace{ax + y^b}, bx - z)$

$t_1x + t_2y + t_3z, \quad t_1, t_2, t_3 \in \mathbb{R}$

$ax + y^b \qquad bx - z$

$x - z$ ✓

$\boxed{b = 1}$

$ax + y$ ✓

$\cancel{b = 0}$

$\cancel{ax + \underline{1}}$

$f(x, y, z) = (ax + y, x - z)$

$\boxed{b = 1, \ a \in \mathbb{R}}$

$$M_f = \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a \\ 0 & -1 & 1 \end{bmatrix}$$

$\Rightarrow \operatorname{rang}(M_f) = 2$

2.2 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x, y) = ax + bxy + cy$

$t_1 x + t_2 y$, $t_1, t_2 \in \mathbb{R}$

$ax + b\underline{xy} + cy$

$\boxed{b = 0}$

$ax + cy$ ✓

$a, c \in \mathbb{R}$

$ax + xy + cy$

$f(x, y) = ax + cy$

$b = 0$, $a, c \in \mathbb{R}$

$$M_f = \begin{bmatrix} a & c \end{bmatrix}$$

$a = 0 \wedge c = 0 \Rightarrow [0 \ \ 0] \Rightarrow \text{rang}(M_f) = 0$

$a \neq 0 \vee c \neq 0 \Rightarrow \text{rang}(M_f) = 1$

2.3 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(\underline{x, y, z}) = (\underline{2^{ay}x + yz^b}, \underline{ax + by + cz})$

$\boxed{t_1 x + t_2 y + t_3 z}$, $t_1, t_2, t_3 \in \mathbb{R}$

$2^{\cancel{ay}} x + y \cancel{z}^{b}$

$ax + by + cz$

cz ✓

$\boxed{a = 0}$, $\boxed{b = 0}$

$x + y$ ✓

$f(x, y, z) = (\underline{x + y}, \underline{cz})$, $a = 0$, $b = 0$, $c \in \mathbb{R}$

$$M_f = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$$

$c = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rang}(M_f) = 1$

$c \neq 0 \Rightarrow \text{rang}(M_f) = 2$

2.4 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = ((\underbrace{ax - b)\, y}, \underbrace{x + ab})$

$t_1 x + t_2 y, \quad t_1, t_2 \in \mathbb{R}$

$(ax - b)y$ $\qquad\qquad\qquad$ $x + ab$

$\cancel{axy} - by$ $\qquad\qquad\qquad$ $x + 0$ ✓

$\boxed{a = 0}$, $b \in \mathbb{R}$

$-by$ ✓

$f(x, y) = (-by, x)$, $\quad a = 0, b \in \mathbb{R}$

$$M_f = \begin{bmatrix} 0 & -b \\ 1 & 0 \end{bmatrix}$$

$b = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \operatorname{rang}(M_f) = 1$

$b \neq 0 \Rightarrow \operatorname{rang}(M_f) = 2$

2.5 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $f(x, y, z) = \left(\frac{ax+b}{bx+a} + y, \sin(bx) + az \right)$

$t_1 x + t_2 y + t_3 z, \quad t_1, t_2, t_3 \in \mathbb{R}$

$\frac{ax+b}{bx+a} + y$

$\sin(bx) + az$

$\boxed{b=0}$

$\frac{ax+0}{0 \cdot x+a} + y$

$\sin 0 + az$

az ✓

$\frac{\cancel{a}x}{\cancel{a}} + y, \boxed{a \neq 0}$ ✓

$x+y$ ✓

$f(x,y,z) = (x+y, az), \quad b=0, \quad a \in \mathbb{R} \setminus \{0\}$

$M_f = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$

$a \neq 0$

$\Rightarrow \operatorname{rang}(M_f) = 2$

2.6 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = \left(\underbrace{x \cdot 5^{(a-1)y+b}}, \underbrace{(\ln b)\, y^2}, \underbrace{ax + cy}\right)$

$$\boxed{\ln A = B \Leftrightarrow A = e^B}$$

$t_1 x + t_2 y, \quad t_1, t_2 \in \mathbb{R}$

$$x \cdot 5^{(a-1)y+b}, \qquad (\ln b)\, \boxed{y^2}, \qquad ax + cy$$

$x \cdot 5^{(a-1)y+1}$

$a - 1 = 0$

$\boxed{a = 1}$

$x \cdot 5^1$

$5x$ ✓

$\ln b = 0$

$b = e^0$

$\boxed{b = 1}$

$0 y^2$

0 ✓

$x + cy$ ✓

$\boxed{c \in \mathbb{R}}$

$f(x,y) = (5x, 0, x + cy)$

$$M_f = \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 1 & c \end{bmatrix} \cdot\left(-\tfrac{1}{5}\right) \sim \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & c \end{bmatrix}$$

$$c = 0 \Rightarrow \begin{bmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rang}(M_f) = 1$$

$$c \neq 0 \Rightarrow \operatorname{rang}(M_f) = 2$$

$f: \mathbb{R}^5 \to \mathbb{R}^2$

(x,y,z,u,v)

$f(x,y,z,u,v) = 0$

3. Za date funkcije ispitati da li su linearne transformacije i za one koje jesu naći jezgro, sliku, rang i matricu.

3.1 $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x,y) = (\underbrace{x-3y}, \underbrace{-2x+6y}, \underbrace{3x-9y})$

- f jeste linearna transformacija jer su sve komponente slike oblika $t_1 x + t_2 y$, $t_1, t_2 \in \mathbb{R}$

- $M_f = \begin{bmatrix} 1 & -3 \\ -2 & 6 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rang}(M_f) = 1 \Rightarrow \operatorname{rang}(f) = 1$

Jezgro: $\boxed{\ker(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}}$!

$f(x,y) = 0$

$(x-3y, -2x+6y, 3x-9y) = 0$

$$\begin{aligned} x - 3y &= 0 \\ -2x + 6y &= 0 \\ 3x - 9y &= 0 \end{aligned}$$

$$\begin{aligned} \boxed{x - 3y = 0} \\ 0 = 0 \\ 0 = 0 \end{aligned} \Rightarrow x = 3y$$

1 x neodređen

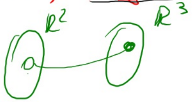
$y = t,\ t \in \mathbb{R},\ x = 3t$

$\ker(f) = \{(3t, t) \mid t \in \mathbb{R}\}$
$= \{(3,1)t \mid t \in \mathbb{R}\}$

Na predavanju: $\ker(f)$ je potprostor $\mathbb{R}^2$, i jedna baza mu je $\{(3,1)\}$

Slika: $\mathrm{Im}(f) = \{ (a,b,c) \in \mathbb{R}^3 \mid \exists (x,y) \in \mathbb{R}^2,\ f(x,y) = (a,b,c) \}$!

$f : \mathbb{R}^2 \to \mathbb{R}^3$



$$f(x,y) = (a,b,c)$$

$$(x-3y,\ -2x+6y,\ 3x-9y) = (a,b,c)$$

$$\begin{aligned} x-3y &= a \\ -2x+6y &= b \\ 3x-9y &= c \end{aligned} \quad \begin{matrix} \cdot 2 \\ \cdot (-3) \end{matrix}$$

$$\begin{aligned} x-3y &= a \\ 0 &= 2a+b \\ 0 &= -3a+c \end{aligned}$$

$$\boxed{\begin{aligned} c &= 3a \\ b &= -2a \end{aligned}} \ !$$

$$\mathrm{Im}(f) = \{ (a,-2a,3a) \mid a \in \mathbb{R} \} = \{ a(1,-2,3) \mid a \in \mathbb{R} \}$$

Na predavanju: $\mathrm{Im}(f)$ je potprostor od $\mathbb{R}^3$, i jedna njena baza je $\{(1,-2,3)\}$.

$\mathrm{rang}(f) = \dim(\mathrm{Im}(f)) = 1$

↳ to smo već znali jer je također $\mathrm{rang}(f) = \mathrm{rang}(M_f)$

3.2 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x,y,z) = (x+y+z, 2x-3y+5z, -2x-7y+z)$

f je linearna transformacija jer su sve komponente slike oblika

$t_1x+t_2y+t_3z$, $t_1,t_2,t_3 \in \mathbb{R}$.

$$M_f = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 5 \\ -2 & -7 & 1 \end{bmatrix} \begin{array}{l} \\ \cdot(-2) \\ \cdot 2 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix} \begin{array}{l} \\ \\ \cdot(-1) \end{array} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \operatorname{rang}(M_f) = 2 \Rightarrow \operatorname{rang}(f) = 2$

JEZGRO: $\ker(f) = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$

$f(x,y,z) = 0$

$(x+y+z, 2x-3y+5z, -2x-7y+z) = 0$

$$\begin{aligned} x+y+z &= 0 \\ 2x-3y+5z &= 0 \quad \cdot(-2) \\ -2x-7y+z &= 0 \quad \cdot 2 \end{aligned}$$

$$\begin{aligned} x+y+z &= 0 \\ -5y+3z &= 0 \\ \cancel{-5y+3z} &\cancel{= 0} \end{aligned}$$

1x neodređen

$$\begin{aligned} x+y &= -z \\ -5y &= -3z \end{aligned}$$

$z = t, t \in \mathbb{R}$

$y = \frac{3}{5}t$

$x = -\frac{8}{5}t$

$\ker(f) = \{(-\frac{8}{5}t, \frac{3}{5}t, t) \mid t \in \mathbb{R}\}$

$= \{(-\frac{8}{5}, \frac{3}{5}, 1)t \mid t \in \mathbb{R}\}$

$\ker(f)$ je potprostor od $\mathbb{R}^3$, baza je

$\{(-\frac{8}{5}, \frac{3}{5}, 1)\}$

SLIKA: $\text{Img}(f) = \{ \underline{(a,b,c) \in \mathbb{R}^3} \mid \exists \underline{(x,y,z) \in \mathbb{R}^3}, \underline{f(x,y,z) = (a,b,c)} \}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x,y,z) = (a,b,c)$

$(x+y+z,\ 2x-3y+5z,\ -2x-7y+z) = (a,b,c)$

$$\begin{aligned} x+y+z &= a \quad \cdot(-2),\ \cdot 2 \\ 2x-3y+5z &= b \\ -2x-7y+z &= c \end{aligned}$$

$$\begin{aligned} x+y+z &= a \\ -5y+3z &= -2a+b \quad \cdot(-1) \\ -5y+3z &= 2a+c \end{aligned}$$

$$\begin{aligned} x+y+z &= a \\ -5y+3z &= -2a+b \\ \boxed{0 = 4a-b+c} \end{aligned}$$

$$\boxed{b = 4a + c}$$

$$\text{Img}(f) = \{ (a, \overbrace{4a+c}^{b}, c) \mid a, c \in \mathbb{R} \}$$

$$= \{ \underline{(1, 4, 0)}\, a + \underline{(0, 1, 1)}\, c \mid a, c \in \mathbb{R} \}$$

$\text{Img}(f)$ je potprostor od $\mathbb{R}^3$ i jedna baza je $\{(1,4,0), (0,1,1)\}$

$\dim(\text{Img}(f)) = 2 = \text{rang}(f)$

4. Neka su linearne transformacije $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ definisane sa $f(x, y) = (2x - y, x + 3y)$ i $g(x, y) = (-x + y, 3x - 2y)$.

4.1 Odrediti kopoziciju $f \circ g$.

4.2 Napisati matrice M_f i M_g linearnih transformacija f i g.

4.3 Naći linearnu transformaciju h koja odgovara matrici $M_f \cdot M_g$ i uporediti je sa $f \circ g$.

4.4 Odrediti f^{-1} i g^{-1} ako postoje.

4.1 $f\circ g(x,y) = f(g(x,y)) = f(-x+y, 3x-2y) = (2\cdot(-x+y) - (3x-2y), -x+y+3(3x-2y))$

$= (-2x+2y-3x+2y, -x+y+9x-6y) = (-5x+4y, 8x-5y)$

4.2. $M_f = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$; $M_g = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

4.3. $M_f \cdot M_g = \underset{2\times 2}{\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}} \cdot \underset{2\times 2}{\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}} = \begin{bmatrix} -5 & 4 \\ 8 & -5 \end{bmatrix} \Rightarrow h(x,y) = (-5x+4y, 8x-5y)$

$\Rightarrow h = f\circ g$

4.4 $M_f = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ $\qquad \mathrm{adj}(M_f) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\det(M_f) = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = 6+1 = 7 \neq 0$ $\qquad M_f^{-1} = \frac{1}{\det(M_f)} \mathrm{adj}(M_f) = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$

$\Rightarrow f^{-1}(x,y) = \left(\frac{3}{7}x + \frac{1}{7}y, -\frac{1}{7}x + \frac{2}{7}y\right)$

$f\circ g$

$$f(x,y) = (2x - y,\ x + 3y)$$

$$g(x,y) = (x + y,\ -x - 2y)$$

$f \circ g$

$$M_f = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \quad M_g = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$M_f \cdot M_g = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & -5 \end{bmatrix}$$

$$f \circ g\,(x,y) = (3x + 4y,\ -2x - 5y)$$

f^{-1}:

$$M_f^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{bmatrix} \Rightarrow f^{-1}(x,y) = \left(\frac{2}{7}x + \frac{1}{7}y,\ -\frac{1}{7}x + \frac{3}{7}y\right)$$

$$\det(M_f) = 6 + 1 = 7 \qquad \operatorname{adj}(M_f) = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$M_g = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\det(M_g) = \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = 2 - 3 = -1 \neq 0$$

$$\operatorname{adj}(M_g) = \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix}$$

$$M_g^{-1} = \frac{1}{\det(M_g)} \operatorname{adj}(M_g) = - \begin{bmatrix} -2 & -1 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$\Rightarrow \quad g^{-1}(x, y) = (2x + y, \ 3x + y)$$

5. Neka su linearne transformacije $f, g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ definisane sa
$f(x, y, z) = (x - 2y, y + z, -2x + y - z)$ i
$g(x, y, z) = (x + 2y + 3z, x - z, 2y + 4z)$.

5.1 Napisati matrice M_f i M_g linearnih transformacija f i g i odrediti njihov rang.

5.2 Odrediti $f \circ g.$, f^{-1}, $ker(g)$ i $Img(g)$.

5.1.

$$M_f = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow rang(M_f) = 3$$

$$\Rightarrow rang(f) = 3$$

$$M_g = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 2 & 4 \end{bmatrix} \overset{-1}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rang(M_g) = 2$$

$$\Rightarrow rang(g) = 2$$

5.2. $f \circ g(x, y, z) = f(g(x, y, z)) = f(x + 2y + 3z, x - z, 2y + 4z) =$ ZA VEŽBU, ALI MOŽE LAKŠE

$$M_f \cdot M_g = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 2 & 3 \\ -1 & -6 & -11 \end{bmatrix} \Rightarrow f \circ g(x, y, z) = (-x + 2y + 5z, x + 2y + 3z, -x - 6y - 11z)$$

$$M_f = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$\det(M_f) = \left|\begin{matrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{matrix}\right| \begin{matrix} 1 & -2 \\ 0 & 1 \\ -2 & 1 \end{matrix} = -1 + 4 - 1 = 2 \neq 0$$

$$\operatorname{adj}(M_f) = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -1 & 3 \\ -2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\Rightarrow M_f^{-1} = \frac{1}{\det(M_f)} \operatorname{adj}(M_f) = \frac{1}{2} \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow f^{-1}(x,y,z) = \left(-x-y-z,\ -x-\frac{1}{2}y-\frac{1}{2}z,\ x+\frac{3}{2}y+\frac{1}{2}z\right)$$

$g: \mathbb{R}^3 \to \mathbb{R}^3, \quad g(x,y,z) = (x+2y+3z,\ x-z,\ 2y+4z)$

JEZGRO: $\ker(g) = \{(x,y,z) \in \mathbb{R}^3 \mid g(x,y,z) = 0\}$

$g(x,y,z) = 0$

$(x+2y+3z,\ x-z,\ 2y+4z) = 0$

$$\begin{aligned} x+2y+3z &= 0 \quad \cdot(-1) \\ x \qquad -z &= 0 \\ 2y+4z &= 0 \end{aligned}$$

$$\begin{aligned} x+2y+3z &= 0 \\ -2y-4z &= 0 \\ 2y+4z &= 0 \quad + \end{aligned}$$

$$\begin{aligned} x+2y+3z &= 0 \\ -2y-4z &= 0 \\ 0 &= 0 \end{aligned}$$

1x neodređen

$$\begin{aligned} x+2y &= -3z \\ 2y &= -4z \end{aligned}$$

$z = t,\ t \in \mathbb{R}$

$y = -2t$

$x = t$

$\ker(g) = \{(t,-2t,t) \mid t \in \mathbb{R}\}$

$= \{t(1,-2,1) \mid t \in \mathbb{R}\}$

$\{(1,-2,1)\}$ – je baza prostora $\ker(g)$ koji je potprostor od $\mathbb{R}^3$

SLIKA: $\operatorname{Img}(g) = \{(a,b,c) \in \mathbb{R}^3 \mid \exists (x,y,z) \in \mathbb{R}^3,\ g(x,y,z) = (a,b,c)\}$

$g(x,y,z) = (a,b,c)$

$(x+2y+3z,\ x-z,\ 2y+4z) = (a,b,c)$

$$\begin{aligned} x+2y+3z &= a \quad \cdot(-1) \\ x \qquad -z &= b \\ 2y+4z &= c \end{aligned}$$

$$\begin{aligned} x+2y+3z &= a \\ -2y-4z &= b-a \\ 2y+4z &= c \quad + \end{aligned}$$

$$\begin{aligned} x+2y+3z &= a \\ -2y-4z &= b-a \\ \boxed{0} &\boxed{= b-a+c} \end{aligned}$$

$\boxed{a = b+c}$

$\operatorname{Img}(g) = \{(b+c, b, c) \mid b,c \in \mathbb{R}\}$

$= \{(1,1,0)b + (1,0,1)c \mid b,c \in \mathbb{R}\}$

$\{(1,1,0),(1,0,1)\}$ baza prostora $\operatorname{Img}(g)$ koji je potprostor od $\mathbb{R}^3$

6. Dati su vektori $\vec{a} = (1, 2, -1)$, $\vec{b} = (3, -1, 1)$ i $\vec{v} = (x, y, z)$. Neka su f, g i h funkcije date sa:

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x, y, z) = \boxed{\vec{a} \times \vec{v} + (\vec{a} \cdot \vec{v}) \cdot \vec{b};}$

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $g(x, y, z) = (y, z)$;

$h : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $h(x, y) = (x - 2y, 2x + y, -y)$.

Dokazati da je funkcija $F = h \circ g \circ f$ linearna transformacija i odrediti njenu matricu.

$$\vec{a} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ x & y & z \end{vmatrix}$$

$$(\vec{a} \cdot \vec{v}) \cdot \vec{b} = (\underbrace{(1, 2, -1) \cdot (x, y, z)}) \cdot (3, -1, 1)$$

$$f(x, y, z) = \vec{a} \times \vec{v} + (\vec{a} \cdot \vec{v}) \cdot \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ x & y & z \end{vmatrix} + (\underbrace{(1, 2, -1) \cdot (x, y, z)}_{\text{BROJ}}) \cdot (3, -1, 1)$$

$$= \begin{vmatrix} 2 & -1 \\ y & z \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ x & z \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ x & y \end{vmatrix} \vec{k} + (x + 2y - z) \cdot (3, -1, 1)$$

$$= \underbrace{(2z + y)\vec{i} - (z + x)\vec{j} + (y - 2x)\vec{k}} + (3(x + 2y - z), -(x + 2y - z), x + 2y - z)$$

$$= (2z + y, -z - x, y - 2x) + (3x + 6y - 3z, -x - 2y + z, x + 2y - z)$$

$$= (3x + 7y - z, -2x - 2y, -x + 3y - z)$$

$$M_f = \begin{bmatrix} 3 & 7 & -1 \\ -2 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix} ; \quad M_g = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_h = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$F = h \circ g \circ f$$

$$\underline{M_F} = M_h \cdot (M_g \cdot M_f) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \cdot \left(\underset{2\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underset{3\times 3}{\begin{bmatrix} 3 & 7 & -1 \\ -2 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix}} \right)$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}_{3\times 2} \begin{bmatrix} -2 & -2 & 0 \\ -1 & 3 & -1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 0 & -8 & 2 \\ -5 & -1 & -1 \\ 1 & -3 & 1 \end{bmatrix}$$

$\Rightarrow F(x,y,z) = (-8y+2z,\ -5x-y-z,\ x-3y+z)$ – jeste lin. fr.

7. Za sledeće funkcije ispitati, odnosno diskutovati po parametrima da li su linearne transformacije i u slučajevima kada jesu naći njihovu matricu, jezgro, sliku i rang.

   7.1 $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(\vec{v}) = \dfrac{\vec{n} \times \vec{v}}{|\vec{n}|}$, $\vec{v} \in \mathbb{R}^3$, $\vec{n} = (-1, 1, 2)$;

7.2 $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $g(\vec{v}) = (\vec{n} \cdot \vec{v}) \cdot \vec{m}$, $\vec{v} \in \mathbb{R}^3$, $\vec{n} = (1, 1, q)$, $\vec{m} = (0, 1, 0)$;

7.3 $h:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $h(\vec{v}) = (\vec{n} \times \vec{v}) \cdot \vec{n} \cdot \vec{m} + 2(\vec{v} \cdot \vec{n}) \cdot \vec{j}$, $\vec{v} \in \mathbb{R}^3$,
$\vec{n} = (1,1,2)$, $\vec{m} = (0,p)$, $\vec{j} = (0,1)$.

8. Linearna transformacija $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ data je sa $f(1,-1) = (-3,6)$ i $f(-2,1) = (2,-4)$. Odrediti $f(x,y)$ i odgovarajuću matricu M_f linearne transformacije f, a zatim naći njen rang. Da li postoji f^{-1}?

$f(1,-1) = (-3,6)$

$f(-2,1) = (2,-4)$

$f(x,y) = (ax+by,\ cx+dy)$

$$\left.\begin{array}{l} f(1,-1) = (a-b,\ c-d) \\ f(1,-1) = (-3,6) \end{array}\right\} \quad \begin{array}{l} a-b=-3 \\ c-d=6 \end{array}$$

$$\left.\begin{array}{l} f(-2,1) = (-2a+b,\ -2c+d) \\ f(-2,1) = (2,-4) \end{array}\right\} \quad \begin{array}{l} -2a+b=2 \\ -2c+d=-4 \end{array}$$

$$\begin{array}{l} a-b=-3 \\ -2a+b=2 \\ \hline -a=-1 \\ \boxed{a=1} \\ \boxed{b=4} \end{array} \qquad \begin{array}{l} c-d=6 \\ -2c+d=-4 \\ \hline -c=2 \\ \boxed{c=-2} \\ \boxed{d=-8} \end{array}$$

$$\boxed{f(x,y) = (x+4y,\ -2x-8y)}$$

$$M_f = \begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix} \cdot 2 \sim \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

$\mathrm{rang}(M_f) = 1$

9. Linearna transformacija $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ data je sa $f(1,-1,1) = (0,2,1)$, $f(2,0,3) = (-2,13,3)$ i $f(-1,2,0) = (-2,5,-1)$. Odrediti $f(x,y,z)$, a zatim naći $ker(f)$ i $Img(f)$.

10. Linearna transformacija $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ data je sa $f(1,-1,0) = (1,0,1)$, $f(1,2,-4) = (0,-1,2)$ i $f(-2,0,3) = (-1,1,0)$. Odrediti $f(x,y,z)$ i odgovarajuću matricu M_f linearne transformacije f, a zatim izračunati $f(-1,3,0)$.

ZA VEŽBU IZ SKRIPTE

Zadatak 12.1; 12.2; 12.7; 12.8 (uzeti da je $g(x, y, z) = (x, y)$)